Conceptual Problems

Exercise 1 *Given matrices A and B,*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$$

Compute $A + B$ and AB.
Find the transpose of the following matrix:

$$C = \begin{bmatrix} 5 & -2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Solve the following system of equations:

$$\begin{aligned} x + 2y &= 5 \\ 3x - y &= 2 \end{aligned}$$

Exercise 2 *Let $A \in \mathbb{C}^{m\times n}$ with columns a_i, and $B \in \mathbb{C}^{p\times n}$ with columns b_i*

$$A = [a_1, a_2, \ldots, a_n], \qquad B = [b_1, b_2, \ldots, b_n],$$

Show that

$$AB^* = a_1 b_1^* + a_2 b_2^* + \ldots a_n b_n^*$$

Exercise 3 *Let $A \in \mathbb{K}^{m\times n}$. Show that $\mathrm{range}(A)$ is a subspace of $\mathbb{K}^m$.*

Exercise 4 *Prove that for α being the angle enclosed by $x, y \in \mathbb{C}^m$, the relation*

$$\cos(\alpha) = \frac{\mathrm{Re}(\langle x, y\rangle)}{\|x\|\|y\|} \tag{1}$$

holds.

Exercise 5 *Let $X = \{x_1, ..., x_n\} \in \mathbb{K}^m$ be a set of non-zero $n \in \mathbb{N}$ orthogonal vectors. Then X is orthogonal if and only if*

$$X^*X = \begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{bmatrix} \tag{2}$$

where $X = [x_1|...|x_n]$.

Exercise 6 *Let B be a 4×4 matrix to which we apply the following operations:*

1. *double column 1,*
2. *halve row 3,*

3. *add row 3 to row 1,*
4. *interchange columns 1and 4,*
5. *subtract row 2 from each of the other rows,*
6. *replace column 4 by column 3,*
7. *delete column 1 (so that the column dimension is reduced by 1).*

(a) Write the result as a product of eight matrices
(b) Write this as a product ABC of three matrices.

Exercise 7 *The Pythagorean theorem asserts that for a set of n orthogonal vectors $\{x_i\}_{i=1}^n$,*

$$\Big\|\sum_{i=1}^{n} x_i\Big\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

(a) Prove this in the case $n = 2$ by an explicit computation of $\|x_1 + x_2\|^2$.
(b) Show that this computation also establishes the general case, by induction.

Exercise 8 *Let $\mathcal{H} = \{A \in \mathbb{C}^{m\times m} \mid A^* = A\}$ denote the set of $m \times m$ Hermitian matrices, and we equip $\mathcal{H}$ with the "vector addition" (denoted $\oplus$)*

$$\oplus : \mathcal{H} \times \mathcal{H} \to \mathcal{H}\ ;\ (A, B) \mapsto [A \oplus B]_{i,j} = A_{i,j} + B_{i,j}$$

and "scalar multiplication" (denoted $\odot$)

$$\odot : \mathbb{K} \times \mathcal{H} \to \mathcal{H}\ ;\ (\lambda, A) \mapsto [\lambda \odot A]_{i,j} = \lambda \cdot A_{i,j}$$

(a) Prove that $(\mathcal{H}, \oplus, \odot)$ is an $\mathbb{R}$ vector space
[Make sure that all vector space properties are proven.]
(b) Is $(\mathcal{H}, \oplus, \odot)$ is a $\mathbb{C}$ vector space? Prove your answer.

Programming Assignment

Exercise 9 *Generate a Julia environment. Go to https://plutojl.org/ and follow the instructions:*

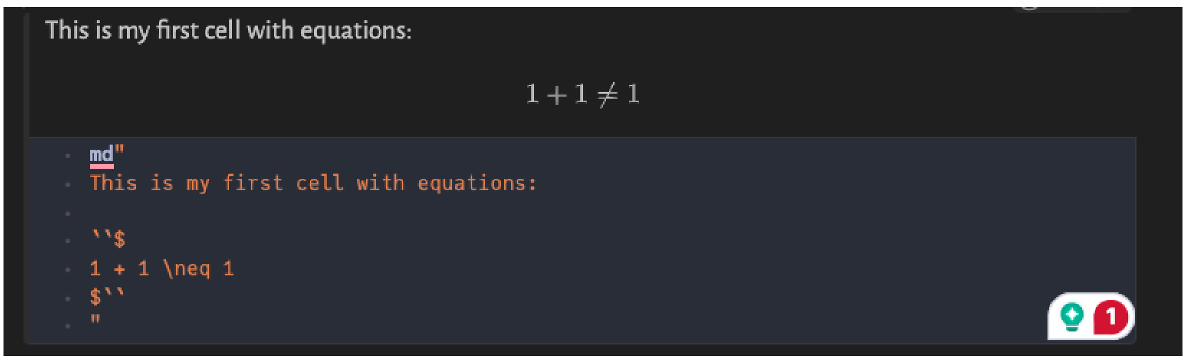
i) Install Julia.

ii) Run Julia: In a terminal, start Julia and then run the VERSION command to verify that you indeed installed the most recent version (`v1.10.4` on August 30, 2024)

iii) Install Pluto (`import Pkg; Pkg.add("Pluto")`)

iv) Run Pluto (`import Pluto; Pluto.run()`)

1. Generate your first markdown cell using LaTeX:



2. Generate a cell that prints a string: