Rensselaer Polytechnic Institute

Computational Linear Algebra (MATH/CSCI 6800)
Final Exam

December 13, 2024

Name: **PUT YOUR NAME HERE**

Exercise	1	2	3	4	5	6	7	8	9	10	11	Total
Points												

Exercise 1 *The Schur factorization:*
(a) Define the Schur factorization for $A \in \mathbb{C}^{m \times m}$.

(b) Prove that every square matrix has a Schur factorization.

Solution:
YOUR ANSWER GOES HERE

Exercise 2 *Courant–Fisher min-max theorem for singular values:*
(a) Complete the Courant–Fisher min-max theorem for singular values:

Let $A \in \mathbb{C}^{m \times n}$*. Then*

$$\sigma_k = \max_{\substack{V \subseteq \mathbb{C}^n \\ \dim(V)=k}} \min_{\substack{v \in V \\ \|v\|_2=1}} \|Av\|_2 \tag{1}$$

and

$$\sigma_{k+1} = ...$$

For the remaining questions:
Let $A \in \mathbb{C}^{m \times n}$ be a rank n matrix with singular values $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n > 0$, and $V \subseteq \mathbb{C}^n$ be a k-dimensional subspace.

(b) Let $\{v_k, ..., v_n\}$ *be the* k*-th to* n*-th right singular vectors of* A*. Prove that*

$$\dim\left(V \cap \operatorname{span}\{v_k, ..., v_n\}\right) \geq 1$$

(c) Show that there exists a vector $v \in V$ *with* $\|v\|_2 = 1$ *such that*

$$\|Av\|_2 \leq \sigma_k.$$

(d) Form this, deduce Eq. (**??**) *in the Courant–Fisher min-max theorem stated above.*

Solution:
YOUR ANSWER GOES HERE

Exercise 3 Let $A \in \mathbb{C}^{n \times n}$ and let R_i be the sum of the absolute values of the non-diagonal entries in the i-th row of A, i.e.,

$$R_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}| \,.$$

Let

$$D(a_{ii}, R_i) \subseteq \mathbb{C}$$

be the closed disc centered at a_{ii} with radius R_i.

Prove that every eigenvalue of A lies within at least one of the discs $D(a_{ii}, R_i)$.

Solution:
YOUR ANSWER GOES HERE

Exercise 4
Consider the matrix

$$\begin{bmatrix} 16 & 4 & 4 & -4 \\ 4 & 10 & 4 & 2 \\ 4 & 4 & 6 & -2 \\ -4 & 2 & -2 & 4 \end{bmatrix}$$

Compute its Cholesky decomposition by hand.

Solution:
YOUR ANSWER GOES HERE

Exercise 5 *The Moore-Penrose inverse:*

(a) Define the Moore-Penrose inverse.

(b) Prove that if A attains an inverse, then

$$A^{-1} = A^{+}$$

(c) Compute the Moore-Penrose inverse of

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$$

Solution:
YOUR ANSWER GOES HERE

Exercise 6 *Induced matrix norms:*
(a) Recall that the vector 1- and 2-norm on $\mathbb{C}^s$ *are defined as*

$$\|x\|_1 = \sum_{i=1}^{s} |x_i| \qquad \text{and} \qquad \|x\|_2 = \sqrt{\sum_{i=1}^{s} x_i^2}.$$

Prove that

$$\|x\|_2 \leq \|x\|_1$$

(b) Prove that

$$\|x\|_1 \leq \sqrt{s}\|x\|_2$$

Hint: Use the Cauchy–Schwarz inequality.

(c) Consider the induced 1- and 2-norm, i.e., for $A \in \mathbb{C}^{m\times n}$

$$\|A\|_1 = \sup_{\substack{x\in\mathbb{C}^n \\ x\neq 0}} \frac{\|Ax\|_1}{\|x\|_1} \qquad \text{and} \qquad \|A\|_2 = \sup_{\substack{x\in\mathbb{C}^n \\ x\neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

Prove that

$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$$

Solution:
YOUR ANSWER GOES HERE

Exercise 7 *The Householder transformation:*
(a) Define the Householder transformation about the hyperplane defined by the normal vector $v \in \mathbb{C}^m$ together with its matrix representation.

(b) Show that the Householder transformation matrix P_v that corresponds to the Householder transformation about the hyperplane defined by the normal vector $v \in \mathbb{C}^m$ fulfills the following properties:

(i)	$P_v^* = P_v$	(iv)	P_v has eigenvalues ± 1
(ii)	$P_v^{-1} = P_v^*$	(v)	$\det(P_v) = -1$
(iii)	$P_v^{-1} = P_v$		

(c) Write a Pseudo-code that computes the QR factorization with Householder. This function takes the matrix $A \in \mathbb{C}^{m\times n}$ as input and returns $R \in \mathbb{C}^{m\times n}$ together with the Householder vectors.
NOTE: You may use MATLAB notation for accessing columns/rows of matrices.

(d) Show that the number of operations in the QR factorization with Householder scales as

$$\#\text{Operations} \in \mathcal{O}(n^2 m)$$

Solution:
YOUR ANSWER GOES HERE

Exercise 8 *Singular value decomposition:*
(a) Define the singular value decomposition of $A \in \mathbb{C}^{m\times n}$.

(b) Compute the singular value decomposition of the matrix

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$$

by hand.

For the remaining exercise:
Numerically, an essential part of the SVD algorithm is the bi-diagonalization by Golub-Kahan or Lawson-Hansen-Chan. For $A \in \mathbb{C}^{m\times n}$, the bi-diagonalization by Golub-Kahan scales as

$$\mathcal{O}(4mn^2 - \frac{4}{3}n^3)$$

whereas the bi-diagonalization using Lawson-Hansen-Chan scales as

$$\mathcal{O}(2mn^2 + 2n^3)$$

(c) Given the matrix $A \in \mathbb{C}^{m\times n}$, *compute the column-row ratio* (m/n) *for which the respective number of flops divided by* n^3 *cross.*

Solution:
YOUR ANSWER GOES HERE

Exercise 9 IEEE 754 single precision format is a 32-bit registry consisting of a one-bit sign, an 8-bit exponent, and a 23-bit mantissa.

(a) What is the range for the exponent with 8 bits (starting from zero)?

For the remaining exercises:
Consider the bias $b = -127$, and reserve the all 0s and all 1s exponents for special characters.

(b) Taking the above into account, what is the range for the exponent with 8 bits?

(c) What is the smallest positive number that can be represented in this format?

(d) What is the largest positive number that can be represented in this format?

(e) What is the machine epsilon in this format?

Solution:
YOUR ANSWER GOES HERE

Exercise 10 *Julia:*
The following Julia code will not run.
Find all syntax errors and list them below, when listing correct code, you will be deducted points.

```
begin

using LinearAlgebra

function hilbert_matrix(n::Int)
    H = [1 / (i + j -1) for i in 1:n, j in 1:n]
    return H
end

function compute_trace(mat::Matrix)
    (m,_) = size(mat)
    trace = 0
    for i in 0 : m
        trace += mat[i,i]
    end
    return trace
end

function int_dim(A::Matrix)
    return compute_trace(A)/opnorm(A, 2)
end

k = 5.0
matrix = hilbert_matrix(k)
id = int_dim(matrix)
display("The intrinsic dimension of the Hilber matrix H($k) is $id')

end
```

Solution:
YOUR ANSWER GOES HERE