Conceptual Problems

Exercise 1. *Suppose* $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_m| \geq 0$ *and* $q_1^T v^{(0)} \neq 0$. *Then show that the iterates of the power method satisfy*

$$\left\| v^{(k)} - (\pm q_1) \right\| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^k \right), \quad \left| \lambda^{(k)} - \lambda_1 \right| = O\left(\left| \frac{\lambda_2}{\lambda_1} \right|^{2k} \right)$$

as $k \to \infty$. *The* $\pm$ *sign means that at each step* k, *one or the other choice of sign is to be taken, and then the indicated bound holds.*

Exercise 2. *Let* $A \in \mathbb{C}^{n \times n}$ *be a Hermitian matrix and define the Rayleigh quotient as:*

$$r(x) = \frac{x^* A x}{x^* x}$$

for any non-zero vector $x \in \mathbb{C}^n$. *Show that the pair* $(r(x_0), x_0)$ *is an eigenpair of* A *iff* x_0 *is a stationary point of* $r(x)$.

Exercise 3. *Let* $A \in \mathbb{C}^{m \times m}$ *be Hermitian. Show that the following statements are equivalent by proving* $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow a$.

(a) The eigenvalues of A *are positive.*

(b) A *is positive definite.*

(c) Δ_k *is positive definite for* $k = 1, \ldots, m$, *where* $\Delta_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}$.

(d) A *has a Cholesky factorization.*

Programming Assignment

To receive full credit for the following programming exercises, please ensure that your code correctly handles any relevant input.

Exercise 4. *Write a Julia function named RQ_iter that performs the Rayleigh quotient iteration introduced in class. The function takes* A, tol, μ *as arguments and returns the tuple* $(eigenval, \mathbf{v}, iter)$ *where eigenval is the maximum eigenvalue and* $\mathbf{v}$ *is the corresponding eigenvector. The argument tol is used to set a stopping condition on the algorithm and iter saves the no. of times the loop has executed before convergence.*

Exercise 5. *Write a Julia function to implement the QR iteration. The function name must be QR_iteration and must take a matrix A as argument. The function must return all the eigenvalues of A (as a list of values). You may assume that A is a matrix with eigenvalues having distinct magnitudes.*

Exercise 6. *Consider the Wilkinson's polynomial defined as follows:*

$$w(x, n) = \Pi_{i=1}^{n}(x - i).$$

1. *Write a Julia function named 'polynomial' that takes* x, n *as input and returns* $w(x, n)$.
2. *Write a function that computes the polynomial coefficients of* $w(x, n)$. ***Hint:*** *Use the package Polynomials. The function name must be 'coefficients' and must take* n *as input and return the coefficients of the polynomial as a single list.*
3. *Write a function that computes the roots given the polynomial coefficients as input. The function name must be 'comp_roots' and must return the roots of the Wilkinson's polynomial as a single list.*
4. *Write a function that visualizes the computed roots in the complex plane given the polynomial coefficients as input. The function name is 'visualize_roots' and the function must return the scatter plot.*

Use the functions you defined above to solve the following exercise:

Consider the perturbation for

$$w(x, 20) + \alpha x^{15}$$

for $\alpha \in \mathcal{N}(0, 1)$, *i.e., randomly drawn from the normal distribution of mean* 0 *and variance* 1. *Draw* 100 *of such perturbations, and compute the roots. Visualize the computed roots in the complex plane.*