F.M. Faulstich **Math 6800: Homework assignment 4** **Due:** 10/03

Conceptual Problems

Exercise 1. *Let $A \in \mathbb{K}^{m\times n}$. Prove that:*

$$\kappa(A^*A) = \kappa(A)^2$$

Exercise 2. *Prove the Eckart–Young–Mirsky theorem for the nuclear norm:*
Let $A \in \mathbb{C}^{m\times n}$ with $A = U\Sigma V^*$ its SVD and rank(A)= r. We define for any $k \in [r]$

$$A_k = \sum_{j=1}^{k} \sigma_j u_j v_j^* = U\Sigma_r V^*$$

with $[\Sigma_r]_{i,i} = \sigma_i$ if $i \leq k$ and zero else. Then

$$\|A - A_k\|_{Tr} = \inf_{\substack{B\in\mathbb{C}^{m\times n}\\ rank(B)\leq k}} \|A - B\|_{Tr} = \sum_{i=k+1}^{r} \sigma_i.$$

Exercise 3. *Prove that the matrix product can be written as a sum of rank-one matrices, i.e.,*

$$\mathbf{AB} = \sum_k \mathbf{A}[:,k]\mathbf{B}[k,:]$$

Programming Assignment

To receive full credit for the following programming exercises, please ensure that your code correctly handles any relevant input. Please submit one Julia file for each exercise (named as HA4_Exno., for eg., the file with code for Exercise 4 should be HA4_Ex4.jl). No test cases or main() function required. Only include Julia functions in each file.

Exercise 4. *Write a Julia function named transform_ bidiagonal that takes a matrix A as input and uses the bi-diagonalization process (i.e., applying a series of Householder transformations to the left and right) to transform A to upper bi-diagonal matrix $B = U_0^T A V_0$. The function returns matrix B.*

Exercise 5. *Implement a Julia function implicit_ qr_ step that takes the matrix B as input and performs one implicit shifted QR iteration on the matrix $B^T B$ (without ever forming $B^T B$ explicitly). Your function should:*

- *Choose a shift λ_k (use Wilkinson's shift).*
- *Perform the first Givens rotation using the first column of $B_{k-1}^T B_{k-1} - \lambda_k I$ and use bulge-chasing to compute U_k and V_k so that $B_k = U_k^T B_{k-1} V_k$ is again upper bi-diagonal.*

Exercise 6. *Write a Julia function compute_SVD that takes a matrix A as input and integrates the previous functions to compute the SVD of A by iteratively applying the QR steps until convergence. The function must return the three matrices U, B, V^T in the same order.*