F.M. Faulstich **Math 6800: Homework assignment 5** **Due:** 10/17

Conceptual Problems

Exercise 1. ***(Nested Approximation Space).*** *We consider $A \in \mathbb{R}^{n\times n}$ a symmetric matrix and Q an orthogonal matrix with columns $q_1, \dots, q_n$. The matrix Q_k is formed by appending the first k column vectors $q_1, \dots, q_k$. We denote*

$$M_k = \lambda_{\max}\big(Q_k^T A Q_k\big), \quad m_k = \lambda_{\min}\big(Q_k^T A Q_k\big),$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the largest and smallest eigenvalues. Prove that

$$M_1 \leq M_2 \leq \cdots \leq M_n = \lambda_{\max}(A), \qquad m_1 \geq m_2 \geq \cdots \geq m_n = \lambda_{\min}(A).$$

Exercise 2. ***(Early Termination)*** *Let $A \in \mathbb{R}^{n\times n}$ be symmetric and let*

$$\mathcal{K}(A, q_1, n) = \operatorname{span}\{q_1,\ Aq_1,\ A^2q_1, \dots, A^{n-1}q_1\}$$

have dimension K. The Lanczos algorithm with starting vector q_1 produces scalars $\{\alpha_i\}_{i\geq 1}$, $\{\beta_i\}_{i\geq 0}$ and orthonormal vectors $\{q_i\}_{i\geq 1}$ satisfying

$$A\, q_i = \beta_{i-1}\, q_{i-1} + \alpha_i\, q_i + \beta_i\, q_{i+1},$$

and is said to terminate at step k if $\beta_k = 0$ and $\beta_i \neq 0$ for all $i < k$. Show that the process terminates in at most K steps in exact arithmetic.

Exercise 3. *We are now going to consider some details about how Krylov methods act in specific situations.*

1. *Given a positive definite matrix A and vector b, prove that if the Lanczos process breaks down at some point (i.e., $\beta_k = 0$ using the notation from Trefethen and Bau) then the subspace $\mathcal{K}_k(A, b)$ contains a solution to the linear system $Ax = b$. In principle, we might be worried that if $\beta_k = 0$ things have gone horribly wrong since we cannot construct the next vector in our orthonormal basis. However, this result shows that in this context everything has actually gone remarkably well.*

2. *Given a non-singular diagonalizable matrix A with at most p distinct eigenvalues and a vector b, show that a solution to $Ax = b$ exists in $\mathcal{K}_k(A, b)$ for some $k \leq p$. In other words, we certainly have a solution in the pth Krylov subspace, though we may find one sooner in some special circumstances.*

Programming Assignment

To receive full credit for the following programming exercises, please ensure that your code correctly handles any relevant input. Please submit one Julia file for each exercise (named as HA5_Exno., for eg., the file with code for Exercise 4 should be HA5_Ex4.jl). No test cases or main() function required. Only include Julia functions in each file.

Exercise 4. *Write a Julia function named 'Arnoldi_ Compute_Hk' that takes a matrix A, a random vector q and an integer kmax as input and computes a $kmax \times kmax$ matrix H_k such that $q^T Aq = H_k$ using Arnoldi process. The function must return the matrix H_k as output.*

Exercise 5. *Write a Julia function named 'Lanczos_ Compute_Hk' that takes a symmetric matrix P, a random vector q and an integer kmax as input and computes a $kmax \times kmax$ tri-diagonal matrix T such that $q^T Aq = T$ using Lanczos process. The function must return the matrix T as output.*

Exercise 6. *Write a Julia function named 'Restart_ Arnoldi' that implements the basic explicit restarting scheme for the Arnoldi method. The function takes a matrix A, a random vector q, an integer kmax (maximum number of Arnoldi steps before restarting) and an integer num_restarts (number of restart cycles to perform) as input. You may select the last Ritz vector in the previous cycle as a new starting vector for the restarting process. The function returns eigenvals (the final Ritz values), eigenvecs (the final Ritz vectors) and H_final (the final Hessenberg matrix from the last restart cycle).*