F.M. Faulstich **Math 6590: Homework assignment 2** **Due:** Friday Feb. 16, 2024.

Exercise 1 *Suppose you have a fair coin (i.e., the probability of heads (H) is the same as tails (T), which is 0.5). The experiment is to flip the coin n times, and we denote the random variable representing the number of heads by X. Note that this is a Bernoulli process, i.e., the individual flips are modeled by a Bernoulli random variable that takes the value 1 (i.e. heads) with probability 0.5, and the value 0 (i.e. tails) with probability 0.5, and*

$$X = \sum_{i=1}^{n} X_i.$$

You are interested in estimating the probability that the number of heads is significantly higher or lower than the expected number, i.e., estimating the tail of the distribution.

a) What is the expected number of heads in n flips, i.e., $\mathbb{E}(X)$?

b) Chernoff bound:

i) Use the Chernoff bound to estimate the probability that the number of heads is at least $(1+\delta)\mathbb{E}[X]$.

ii) Calculate this probability for $n = 100$ and $\delta = 0.1$. Interpret your results.

Exercise 2 *Consider*

$$g(u) = \gamma u + \ln\left(1 + \gamma - \gamma e^{u}\right)$$

with $\gamma = \frac{a}{b-a}$. Prove that

$$g''(u) \leq \frac{1}{4} \qquad \text{for } u > 0$$

Exercise 3

a) Implement the Girard–Hutchinson estimator

b) Experiment with the Girard–Hutchinson estimator. Choose $\boldsymbol{\omega}_i$ to be:

i) Gaussian: The entries of of each $\boldsymbol{\omega}_i$ are i.i.d. $\mathcal{N}(0,1)$

ii) Rademacher: The entries of of each $\boldsymbol{\omega}_i$ are i.i.d. Rademacher

iii) Sphere: The entries of of each $\boldsymbol{\omega}_i$ are i.i.d. uniformly sampled from a sphere of radius $\sqrt{n}$, i.e., $\boldsymbol{\omega}_i \sim \mathcal{U}\{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x}^\top \mathbf{x} = n\}$ where $\mathcal{U}$ is the uniform distribution.

Generate $\mathbf{A} \in \mathbb{R}^{1000\times1000}$ with eigenvalues uniformly distributed between 0.9 and 1.1 with a (Haar orthogonal) random matrix of eigenvectors. Investigate the variance $\mathbb{V}(\boldsymbol{\omega}^* \mathbf{A} \boldsymbol{\omega})$ for the cases i)–iii), what do you observe, and which random variable would you use in computations?
Note: You may find it useful to normalize the Variance by $\operatorname{Tr}(\mathbf{A})$

c) Implement the remaining trace estimators introduced in class:

ii) The HUTCH++ *estimator*

iii) The XTRACE *Estimator*

iv) The XNysTrace *Estimator*

d) Fix the random seed and compute that Girard–Hutchinson, XTrace *and* XNysTrace *are indeed exchangeable, whereas* Hutch++ *is not.*

e) Reproduce the Figure 1 in [1]

References

[1] Ethan N Epperly, Joel A Tropp, and Robert J Webber. Xtrace: Making the most of every sample in stochastic trace estimation. *SIAM Journal on Matrix Analysis and Applications*, 45(1):1–23, 2024.