F.M. Faulstich **Math 6590: Homework assignment 3** **Due:** Monday Mar. 18, 2024.

Exercise 1 *Let $\mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d}$ be fixed and $\mathbb{R}^{n_1 \times ... \times n_d}$ be the set of all mappings $\mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d} \to \mathbb{R}$. Define an addition $\oplus$ on $\mathbb{R}^{n_1 \times ... \times n_d}$ as*

$$(\mathbf{X} \oplus \mathbf{Y})[i_1, ..., i_d] = X[i_1, ..., i_d] + Y[i_1, ..., i_d],$$

for all $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n_1 \times ... \times n_d}$. The tensor $\mathbf{0} \in \mathbb{R}^{n_1 \times ... \times n_d}$ is defined elementwise via $\mathbf{0}[i_1, ..., i_d] = 0$ for all $i_1, ..., i_d$. Define a scalar multiplication $\otimes$, such that for every scalar $\alpha \in \mathbb{R}$ and $\mathbf{X} \in \mathbb{R}^{n_1 \times ... \times n_d}$

$$(\alpha \otimes \mathbf{X})[i_1, ..., i_d] = \alpha \cdot (X[i_1, ..., i_d]).$$

Show that $\mathbb{R}^{n_1 \times ... \times n_d}$ forms a real vector space.

Consider the third order tensor

$$\mathbf{A} = \left[\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}\right] \in \mathbb{R}^{4 \times 3 \times 2}$$

Exercise 2 *Perform the following tensor-vector contractions:*

$$(a)\ \mathbf{A} *_{3,1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad (b)\ \mathbf{A} *_{3,1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (c)\ \mathbf{A} *_{3,1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Exercise 3 *Perform the following tensor contractions:*

$$(a)\mathbf{A} *_{(2,3),(2,1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (b)\mathbf{A} *_{(1,3),(2,1)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

Exercise 4 *In this exercise, you will numerically scrutinize the different sketching techniques introduced in the lectures.*

a) Implement the Gaussian Embeddings, and provide your code.
b) Implement the SRTT, and provide your code.
c) Implement the SSE, and provide your code.
d) Perform timing comparison of
   - *Construction: The time required to generate the sketching matrix S.*
   - *Vector apply. The time to apply the sketch to a single vector*
   - *Matrix apply. The time to apply the sketch to an n × 200 matrix*

   for $n = 10^6$, $d = 400$, for SRTT with DCT and SSE with $\zeta = 8$.
   Report your results in a table similar to the one we have seen in Lecture 8 [a 3×3 table].
e) Apply sketch-and-solve to a least-squares problem of size 10,000 by 100 with condition number 10^8 and residual norm 10^{-4}. Report the residual norm and the forward error for sketch-and-solve and compare them with results from a direct solver in MATLAB (i.e. "backslash" operator).
f) Implement the iterative sketching,and provide your code.
g) Perform a comparison study, providing **at least** *a plot as shown in class plotting the forward error of the direct, sketch-and-solve, and iterative sketching against each other.*