

F.M. Faulstich **Math 6590: Homework assignment 4 Due:** Wednesday 04/10/2024.

**Exercise 1** Consider the CP decomposition of  $\mathbf{X} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , i.e., the representation of  $\mathbf{X}$  as a sum of elementary tensors

$$\mathbf{X} = \sum_{p=1}^r \mathbf{v}_{1,p} \otimes \dots \otimes \mathbf{v}_{d,p} = \sum_{p=1}^r \bigotimes_{i=1}^d \mathbf{v}_{i,p}$$

for  $\mathbf{v}_{i,p} \in \mathbb{R}^{n_i}$ . Define the matrices

$$\mathbf{V}_i = [\mathbf{v}_{i,1} | \dots | \mathbf{v}_{i,r}] \quad (1)$$

and show that

$$\mathbf{X}[i_1, \dots, i_d] = \sum_{p=1}^r \mathbf{V}_1[i_1, p] \mathbf{V}_2[i_2, p] \dots \mathbf{V}_d[i_d, p] \quad (2)$$

**Exercise 2** Consider the sequence

$$\mathbf{X}_n = n \left( \mathbf{u} + \frac{1}{n} \mathbf{v} \right) \otimes \left( \mathbf{u} + \frac{1}{n} \mathbf{v} \right) \otimes \left( \mathbf{u} + \frac{1}{n} \mathbf{v} \right) - n \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$$

with  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ ,  $\|\mathbf{v}\| = \|\mathbf{u}\| = 1$  and  $\langle \mathbf{v}, \mathbf{u} \rangle \neq 1$ .

i) Show that  $\mathbf{X}_n \in \mathcal{M}_{\leq 2}$  for all  $n \in \mathbb{N}$

ii) Show that

$$\lim_{n \rightarrow \infty} \mathbf{X}_n = \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \notin \mathcal{M}_{\leq 2}$$

**Exercise 3** In this exercise, you will implement and compare the different higher-order SVDs introduced in class.

i) Implement the Truncated Higher-Order SVD (THOSVD)

Input: Target tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , target rank  $r = (r_1, \dots, r_d)$

Output: Core tensor  $\mathbf{C} \in \mathbb{R}^{r_1 \times \dots \times r_d}$ , basis matrices  $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$  for  $1 \leq k \leq d$

ii) Implement the Sequential Truncated Higher Order SVD (STHOSVD)

Input: Target tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , target rank  $r = (r_1, \dots, r_d)$

Output: Core tensor  $\mathbf{C} \in \mathbb{R}^{r_1 \times \dots \times r_d}$ , basis matrices  $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$  for  $1 \leq k \leq d$

iii) Implement the Randomized Sequential Truncated Higher Order SVD (R-STHOSVD)

Input: Target tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , target rank  $r = (r_1, \dots, r_d)$ , oversampling parameter  $p \in \mathbb{N}$

Output: Core tensor  $\mathbf{C} \in \mathbb{R}^{r_1 \times \dots \times r_d}$ , basis matrices  $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$  for  $1 \leq k \leq d$

iv) Implement the sketched Sequential Truncated Higher Order SVD (sketched-STHOSVD)

Input: Target tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , target rank  $r = (r_1, \dots, r_d)$ , oversampling parameter  $\ell = (\ell_1, \dots, \ell_d)$

Output: Core tensor  $\mathbf{C} \in \mathbb{R}^{r_1 \times \dots \times r_d}$ , basis matrices  $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$  for  $1 \leq k \leq d$

v) Implement the sub-sketch Sequential Truncated Higher Order SVD (sub-sketch-STHOSVD)  
 Input: Target tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$ , target rank  $r = (r_1, \dots, r_d)$ , oversampling parameter  $\ell = (\ell_1, \dots, \ell_d)$ , power  $q \in \mathbb{N}$   
 Output: Core tensor  $\mathbf{C} \in \mathbb{R}^{r_1 \times \dots \times r_d}$ , basis matrices  $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$  for  $1 \leq k \leq d$

vi) Run experiment 1 outlined in class:

Consider the Hilbert tensor  $\mathbf{A} \in \mathbb{R}^{n_1 \times \dots \times n_d}$  defined as

$$\mathbf{A}[i_1, \dots, i_d] = \frac{1}{i_1 + \dots + i_d}, \quad 1 \leq i_j \leq n_j, \quad 1 \leq j \leq d$$

Set the tensor parameters:

- $d = 5$
- $n_j = 25$ ,  $j = 1, 2, \dots, d$
- Target rank is  $(r, r, r, r, r)$ , where  $r \in \llbracket 1, 25 \rrbracket$

Choose the computational parameters:

- Oversampling  $p = 5$
- $l_i = r_i + 2$ ,  $i = 1, 2, \dots, d$
- Power parameter  $q = 1$

Reproduce the plots shown in classes (lecture 17)

vii) Now consider a Hilbert tensor  $\mathbf{A} \in \mathbb{R}^{500 \times 500 \times 500}$ . Run experiments targeting the ranks

$$(i \cdot 10, i \cdot 10, i \cdot 10) \quad \text{for } i \in \llbracket 10 \rrbracket$$

Report the CPU time in seconds and the relative error. Reproduce the numbers shown in class (lecture 17).

viii) Run experiment 2 outlined in class:

Consider the sparse tensor  $\mathbf{A} \in \mathbb{R}^{200 \times 200 \times 200}$ :

$$\mathbf{A} = \sum_{i=1}^{10} \frac{\gamma}{i^2} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i + \sum_{i=11}^{200} \frac{1}{i^2} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i,$$

where  $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i \in \mathbb{R}^{200}$  are sparse vectors with 5% nonzeros each (generated using the `sprand` command in MATLAB)

Set the tensor parameters:

- $\gamma = 2, 10, 200$
- $n_j = 25$ ,  $j = 1, 2, \dots, d$
- Target rank is  $(r, r, r)$ , where  $r \in \llbracket 20, 100 \rrbracket$

Choose the computational parameters:

- Oversampling  $p = 5$
- $l_i = r_i + 2$ ,  $i = 1, 2, \dots, d$
- Power parameter  $q = 1$

Reproduce the plots shown in class (lecture 17).