Multi-linear Algebra
Lecture 13

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Idea of tensor products

Consider

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \qquad \text{and} \qquad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$$

Then

$$\mathbf{v}_1 \otimes \mathbf{v}_1 = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} & 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathbf{v}_1 \otimes \mathbf{v}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{v}_2 \otimes \mathbf{v}_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathbf{v}_2 \otimes \mathbf{v}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Tensor product space

Observation:
The tensor products

$$\mathbf{v}_i \otimes \mathbf{v}_j \qquad \text{for } i, j \in \{1, 2\}$$

for a basis of $\mathbb{R}^{2\times 2}$

Can we formalize this idea?

The algebraic tensor product space

Definition (algebraic tensor product)

Let U, V be Hilbert spaces over the same field $\mathbb{F}$. For any $\mathbf{u} \in U$ and $\mathbf{v} \in V$, the conjugate bilinear form

$$\mathbf{u} \otimes \mathbf{v} \ : \ U \times V \to \mathbb{F} \ ; \ (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \to \langle \mathbf{u}, \tilde{\mathbf{u}} \rangle_U \cdot \langle \mathbf{v}, \tilde{\mathbf{v}} \rangle_V$$

is called the tensor product of $\mathbf{u}$ and $\mathbf{v}$. Elements that can be constructed like this are called *elementary tensors*. The linear span (using the canonical addition) of all elementary tensors

$$U \otimes_a V := \operatorname{span} \{ \mathbf{u} \otimes \mathbf{v} \mid \mathbf{u} \in U, \ \mathbf{v} \in V \}$$

is called the algebraic tensor product of U and V.

Tensor interpretation

- A tensor can be interpreted as a linear map:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

then we may also define $\mathbf{v}$ through

$$\mathbf{v}(\mathbf{u}) = v_1\, u_1 + v_2\, u_2$$

which is a linear map of $\mathbb{F}^2$.

Inner product structure

Definition (skalar product)

For every χ, $\tilde{\chi} \in U \otimes_a V$, the induced scalar product $\langle \chi, \tilde{\chi} \rangle_{U \otimes V}$ is defined via any finite linear representation

$$\chi = \sum_i \mathbf{u}_i \otimes \mathbf{v}_i \qquad \mathbf{u}_i \in U, \ \mathbf{v}_i \in V$$

$$\tilde{\chi} = \sum_j \mathbf{u}_j \otimes \mathbf{v}_j \qquad \mathbf{u}_j \in U, \ \mathbf{v}_j \in V$$

as

$$\langle \chi, \tilde{\chi} \rangle_{U \otimes V} := \sum_{i,j} \langle \mathbf{u}_i, \tilde{\mathbf{u}}_j \rangle_U \cdot \langle \mathbf{v}_i, \tilde{\mathbf{v}}_j \rangle_V$$

This induced scalar product on $U \otimes_a V$ is well defined, i.e. does not depend on the linear representations chosen.

Remark on algebraic tensor product spaces

- In finite dimension, $U \otimes_a V$ equipped with the induced scalar product is itself a Hilbert space.
- In general $U \otimes_a V$ may not be complete.

Definition (tensor product)

The closure of the algebraic tensor product of U, V, i.e.,

$$U \otimes V = \overline{U \otimes_a V}$$

with respect to the norm induced by $\langle \cdot, \cdot \rangle_{U \otimes V}$ is called the *topological tensor product* or simply *tensor product*.

Properties

- Tensor basis:
  If $\{\mathbf{u}_i \mid 1 \leq i \leq \dim(U)\}$ and $\{\mathbf{v}_i \mid 1 \leq i \leq \dim(V)\}$ are orthonormal bases of U and V, respectively, then $\{\mathbf{u}_i \otimes \mathbf{v}_j \mid 1 \leq i \leq \dim(U),\ 1 \leq j \leq \dim(V)\}$ is an orthonormal basis of $U \otimes V$. Therefore the dimension of $U \otimes V$ is $\dim(V) \cdot \dim(U)$.
- Tensor subspaces:
  Let $\tilde{U} \subseteq U$ and $\tilde{V} \subseteq V$ be subspaces of U and V, and $\tilde{U}^{\perp}$ and $\tilde{V}\perp$ be the corresponding orthogonal complements then

  $$U \otimes V = (\tilde{U} \otimes \tilde{V}) \oplus (\tilde{U}^{\perp} \otimes \tilde{V}) \oplus (\tilde{U} \otimes \tilde{V}^{\perp}) \oplus (\tilde{U}^{\perp} \otimes \tilde{V}^{\perp})$$

  is an orthogonal decomposition of $U \otimes V$. In particular $\tilde{U} \otimes \tilde{V}$ is a subspace of $U \otimes V$.

Multiple tensor products

Going back to

$$\mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{R}^2 \qquad \text{and} \qquad \mathbf{u} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{R}^2$$

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Then

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How about

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We define this elementwise

$$[\mathbf{v} \otimes \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{v} \otimes \mathbf{u}]_{i,j,k,l,m} = v_i u_j v_k v_l u_m$$

Multiple Algebraic Tensor Product

Definition

For any $\mathbf{u}_1 \in U_1$,..., $\mathbf{u}_d \in U_d$ the conjugate bilinear form

$$\begin{aligned}\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d :\ & U_1 \times ... \times U_d \to \mathbb{F} \\ & (\tilde{\mathbf{u}}_1, ..., \tilde{\mathbf{u}}_d) \to \langle \mathbf{u}_1, \tilde{\mathbf{u}}_1 \rangle_{U_1} \cdots \langle \mathbf{u}_d, \tilde{\mathbf{u}}_d \rangle_{U_d}\end{aligned}$$

is called the tensor product of $\mathbf{u}_1, ..., \mathbf{u}_d$, and elements which can be constructed in this way are called *elementary tensors*. The linear span (using the canonical addition) of all elementary tensors

$$U_1 \otimes \cdot \otimes U_d = \operatorname{span}\{\mathbf{u}_1 \otimes \cdots \otimes \mathbf{u}_d \mid \mathbf{u}_i \in U_i\}$$

is called the algebraic tensor product of $U_1, ..., U_d$. The number d of involved Hilbert spaces is referred to as the order of the tensor space.

The tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$

- We will mostly concern ourselfs with tensors in $\mathbb{R}^{n_1 \times \ldots \times n_d}$
- A tensor is nothing but a map that maps integers to numbers
  Example:
  Let $\mathbb{N}_k = \{1, ..., k\}$. Then

$$\mathbf{x} : \mathbb{N}_k \to \mathbb{R} \; ; \; i \mapsto \mathbf{x}[i]$$

  describes a vector $\mathbf{x} \in \mathbb{R}^k$.

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$$\mathbf{x} : \mathbb{N}_k \to \mathbb{R} \;;\; i \mapsto \mathbf{x}[i]$$

describes a vector $\mathbf{x} \in \mathbb{R}^k$. The map

$$\mathbf{A} : \mathbb{N}_m \times \mathbb{N}_n \to \mathbb{R} \;;\; (i, j) \mapsto \mathbf{A}[i, j]$$

describes a matrix.

The tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$

- The map

$$\boldsymbol{\chi} : \mathbb{N}_{n_1} \times \mathbb{N}_{n_d} \; ; \; (i_1, ..., i_d) \mapsto \boldsymbol{\chi}[i_1, ..., i_d]$$

  defines the tensor $\boldsymbol{\chi} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$. The number n_i is the ith dimension of $\boldsymbol{\chi}$, $\mathbb{N}_{n_1} \times \mathbb{N}_{n_d}$ its index set and d its order.
- Tensors are a generalization of vectors and matrices:
  Tensors of order one are vectors
  Tensors of order two are matrices
- Linear algebra is a special case of multi-linear algebra

The tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$

Proposition

Each space $\mathbb{R}^{n_1 \times \ldots \times n_d}$ can be expressed as the d-fold tensor product of order one tensors

$$\mathbb{R}^{n_1 \times \ldots \times n_d} = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d} = \bigotimes_{i-1}^{d} \mathbb{R}^{n_i}$$

In particular, every tensor $\boldsymbol{\chi} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ can be expressed as a linear combination elementary tensors, i.e.,

$$\boldsymbol{\chi} = \sum_j \mathbf{x}_{1,j} \otimes \cdots \otimes \mathbf{x}_{d,j} \qquad \mathbf{x}_{i,j} \in \mathbb{R}^{n_i}$$

Frobenius scalar product

Let the real coordinate spaces $\mathbb{R}^{n_i}$ be equipped with the canonical scalar product.

Frobenius scalar product

Let the real coordinate spaces $\mathbb{R}^{n_i}$ be equipped with the canonical scalar product. Then, the unique induced scalar product on $R^{n_1 \times \ldots \times n_d}$, i.e. the scalar product for which

$$\left\langle \bigotimes_{i=1}^{d} \mathbf{x}_i, \bigotimes_{i=1}^{d} \mathbf{y}_i \right\rangle = \prod_{i=1}^{d} \langle \mathbf{x}_i, \mathbf{y}_i \rangle$$

holds for all elementary tensors with $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{n_i}$, is the Frobenius scalar product defined as

$$\begin{aligned} \langle \cdot, \cdot \rangle_F : \mathbb{R}^{n_1 \times \ldots \times n_d} \times \mathbb{R}^{n_1 \times \ldots \times n_d} &\to \mathbb{R} \\ (\mathbf{X}, \mathbf{Y}) &\mapsto \sum_{i_1, \ldots, i_d} \mathbf{X}[i_1, ..., i_d] \mathbf{Y}[i_1, .., i_d] \end{aligned}$$

Frobenius norm

The induced Frobenius norm is

$$\|\mathbf{X}\|_F = \sqrt{\langle \mathbf{X}, \mathbf{X} \rangle_F} = \sqrt{\sum_{i_1,...,i_d} \mathbf{X}[i_1, ..., i_d]^2}$$

which is a crossnorm, meaning that

$$\|\mathbf{x}_1 \otimes \cdots \otimes \mathbf{x}_d\|_F = \prod_{i=1}^{d} \|\mathbf{x}_i\|$$

holds for all elementary tensors with $\mathbf{x}_i \in \mathbb{R}^{n_i}$.

Vectorization

Definition

Vectorization Given the bijection $\varphi : \mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d} \to \mathbb{N}_{m_{\text{vec}}}$ with $\mathbb{N}_{m_{\text{vec}}} = n_1 \cdots n_d$ the mapping

$$\text{Vec} : \mathbb{R}^{n_1 \times ... \times n_d} \to \mathbb{R}^{m_{\text{vec}}} \ ; \ \mathbf{X} \mapsto \text{Vec}(\mathbf{X})$$

with

$$\text{Vec}(\mathbf{X})[i] = \mathbf{X}[\phi^{-1}(i)]$$

Example: We choose φ to be the lexicographical ordering

$$\varphi : \mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d} \to \mathbb{N}_{m_{\text{vec}}} \ ; \ (i_1, ..., i_d) \mapsto 1 + \sum_{k=1}^{d} (i_k - 1) \prod_{\ell < k} n_\ell$$

Matricization

Definition

Given the tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$, let $\Lambda \subseteq \{1, ..., d\}$ denote a subset of the modes and let Λ^c be its compliment. We define $m_1 = \prod_{i \in \Lambda} n_i$ and $m_2 = \prod_{j \in \Lambda^c}$. Given a bijection

$$\phi : \mathbb{N}_{n_1} \times \cdots \mathbb{N}_{n_d} \to \mathbb{N}_{m_1} \times \mathbb{N}_{m_2}$$
$$(i_1, ..., i_d) \mapsto (\phi_1(i_k \mid k \in \Lambda), \phi_2(i_\ell \mid \ell \in \Lambda^c))$$

The map

$$\mathrm{MAT}_\Lambda : \mathbb{R}^{n_1 \times \cdots n_d} \to \mathbb{R}^{m_1 \times m_2} \ ; \ \mathbf{X} \mapsto \mathrm{MAT}_\Lambda(\mathbf{X})$$

where

$$\mathrm{MAT}_\Lambda(\mathbf{X})[i, j] = \mathbf{X}(\phi^{-1}(i, j))$$

is called the Λ-matricization.