Multi-linear Algebra
– Tensor diagrams –
Lecture 14

F. M. Faulstich

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The tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$

- The map

$$\boldsymbol{\chi} : \mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_d} \;;\; (i_1, \ldots, i_d) \mapsto \boldsymbol{\chi}[i_1, \ldots, i_d]$$

  defines the tensor $\boldsymbol{\chi} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$. The number n_i is the ith dimension of $\boldsymbol{\chi}$, $\mathbb{N}_{n_1} \times \ldots \times \mathbb{N}_{n_d}$ its index set and d its order.
- Tensors are a generalization of vectors and matrices:
  Tensors of order one are vectors
  Tensors of order two are matrices
- Linear algebra is a special case of multi-linear algebra

Tensor diagrams

Tensor diagrams

What can we do with tensor diagrams?

The tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$

Proposition

Each space $\mathbb{R}^{n_1 \times \ldots \times n_d}$ can be expressed as the d-fold tensor product of order one tensors

$$\mathbb{R}^{n_1 \times \ldots \times n_d} = \mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_d} = \bigotimes_{i-1}^{d} \mathbb{R}^{n_i}$$

In particular, every tensor $\boldsymbol{\chi} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ can be expressed as a linear combination elementary tensors, i.e.,

$$\boldsymbol{\chi} = \sum_j \mathbf{x}_{1,j} \otimes \cdots \otimes \mathbf{x}_{d,j} \qquad \mathbf{x}_{i,j} \in \mathbb{R}^{n_i}$$

Tensor product in diagrams

Tensor contractions

Let $\mathbf{X} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ and $\mathbf{Y} \in \mathbb{R}^{m_1 \times \ldots \times m_d}$ be two tensors of order d and e, respectively. The contraction $\mathbf{X} *_{\ell,k} \mathbf{Y}$ of the ℓ-th mode of $\mathbf{X}$ with the k-th mode of $\mathbf{Y}$ is defined elementwise as

$$\mathbf{X} *_{\ell,k} \mathbf{Y}[i_1, ..., i_{\ell-1}, i_{\ell+1}, ..., i_d, j_1, ..., j_{k-1}, j_{k+1}, ..., j_e]$$
$$= \sum_{p=1}^{n_\ell} \mathbf{X}[i_1, ..., i_{\ell-1}, p, i_{\ell+1}, ..., i_d] \cdot Y[j_1, ..., j_{k-1}, p, j_{k+1}, ..., j_e].$$

Example

Consider

$$\mathbf{A} = \left[\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \right] \in \mathbb{R}^{3\times 2\times 2}$$

Then

$$\mathbf{A} *_{2,1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

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$$\mathbf{A} *_{(1,2),(1,2)} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{bmatrix}$$

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$$\mathbf{A} *_{(1,2),(1,2)} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \sum_{1,2} \left[\begin{bmatrix} 1 & 4 \\ 8 & 10 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 10 & 12 \\ 7 & 0 \end{bmatrix} \right] = \begin{bmatrix} 30 \\ 36 \end{bmatrix}$$

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$$\mathbf{A} *_{(1,2),(1,2)} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \sum_{1,2} \left[\begin{bmatrix} 1 & 4 \\ 8 & 10 \\ 7 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 6 \\ 10 & 12 \\ 7 & 0 \end{bmatrix} \right] = \begin{bmatrix} 30 \\ 36 \end{bmatrix}$$

$$\mathbf{A} *_{(1,3),(1,2)} \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 28 \\ 38 \end{bmatrix}$$

Tensor contraction in diagrams

Frobenius scalar product

Let the real coordinate spaces $\mathbb{R}^{n_i}$ be equipped with the canonical scalar product.

Frobenius scalar product

Let the real coordinate spaces $\mathbb{R}^{n_i}$ be equipped with the canonical scalar product. Then, the unique induced scalar product on $R^{n_1 \times \ldots \times n_d}$, i.e. the scalar product for which

$$\left\langle \bigotimes_{i=1}^{d} \mathbf{x}_i, \bigotimes_{i=1}^{d} \mathbf{y}_i \right\rangle = \prod_{i=1}^{d} \langle \mathbf{x}_i, \mathbf{y}_i \rangle$$

holds for all elementary tensors with $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^{n_i}$, is the Frobenius scalar product defined as

$$\langle \cdot, \cdot \rangle_F : \mathbb{R}^{n_1 \times \ldots \times n_d} \times \mathbb{R}^{n_1 \times \ldots \times n_d} \to \mathbb{R}$$
$$(\mathbf{X}, \mathbf{Y}) \mapsto \sum_{i_1, \ldots, i_d} \mathbf{X}[i_1, ..., i_d] \mathbf{Y}[i_1, .., i_d]$$

Example

$$\left\langle \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \right\rangle$$

Example

$$\left\langle \begin{bmatrix} 1 & 2 \\ 6 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} \right\rangle = \sum_{1,2} \begin{bmatrix} 2 & 10 \\ 6 & 0 \end{bmatrix} = 18$$

$$\left\langle \left[\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \right], \left[\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 2 & 1 \\ 4 & 2 \end{bmatrix} \right] \right\rangle = \sum_{1,2,3} \left[\begin{bmatrix} 0 & 2 \\ 8 & 15 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 9 \\ 10 & 6 \\ 28 & 16 \end{bmatrix} \right] = 96$$

Scalar product in diagrams

Vectorization

Definition

Vectorization Given the bijection $\varphi : \mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d} \to \mathbb{N}_{m_{\text{vec}}}$ with $\mathbb{N}_{m_{\text{vec}}} = n_1 \cdots n_d$ the mapping

$$\text{Vec} : \mathbb{R}^{n_1 \times ... \times n_d} \to \mathbb{R}^{m_{\text{vec}}} \ ; \ \mathbf{X} \mapsto \text{Vec}(\mathbf{X})$$

with

$$\text{Vec}(\mathbf{X})[i] = \mathbf{X}[\phi^{-1}(i)]$$

Example: We choose φ to be the lexicographical ordering

$$\varphi : \mathbb{N}_{n_1} \times ... \times \mathbb{N}_{n_d} \to \mathbb{N}_{m_{\text{vec}}} \ ; \ (i_1, ..., i_d) \mapsto 1 + \sum_{k=1}^{d} (i_k - 1) \prod_{\ell < k} n_\ell$$

Vectorization with diagrams

Matricization

Definition

Given the tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$, let $\Lambda \subseteq \{1, ..., d\}$ denote a subset of the modes and let Λ^c be its compliment. We define $m_1 = \prod_{i \in \Lambda} n_i$ and $m_2 = \prod_{j \in \Lambda^c}$. Given a bijection

$$\begin{aligned} \phi : \mathbb{N}_{n_1} \times \cdots \mathbb{N}_{n_d} &\to \mathbb{N}_{m_1} \times \mathbb{N}_{m_2} \\ (i_1, ..., i_d) &\mapsto (\phi_1(i_k \mid k \in \Lambda), \phi_2(i_\ell \mid \ell \in \Lambda^c)) \end{aligned}$$

The map

$$\mathrm{MAT}_\Lambda : \mathbb{R}^{n_1 \times \cdots n_d} \to \mathbb{R}^{m_1 \times m_2} \ ; \ \mathbf{X} \mapsto \mathrm{MAT}_\Lambda(\mathbf{X})$$

where

$$\mathrm{MAT}_\Lambda(\mathbf{X})[i, j] = \mathbf{X}(\phi^{-1}(i, j))$$

is called the Λ-matricization.

Matricization with diagrams