Eigenvalue Computations
Lecture 20

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Eigenvalue Problem

Given $\mathbf{A} \in \mathbb{C}^{m \times m}$. Then,

i) we call $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^m$ eigenvector of $\mathbf{A}$

ii) we call $\lambda \in \mathbb{C}$ eigenvalue

if

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

Intuitively:

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Intuitively:

The action of $\mathbf{A}$ on a subspace $S \subseteq \mathbb{C}^m$ mimics a scalar multiplication. The subspace S is then called an eigenspace, and any nonzero $\mathbf{x} \in S$ is an eigenvector.

We denote the set of all eigenvalues (the spectrum) of $\mathbf{A}$ by $\Lambda(\mathbf{A}) \subset \mathbb{C}$

Eigenvalue Decomposition

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An eigenvalue decomposition of $\mathbf{A} \in \mathbb{C}^{m \times m}$ is a factorization

$$\mathbf{A} = \mathbf{X}\boldsymbol{\Lambda}\mathbf{X}^{-1}$$

where $\mathbf{X}$ is non-singular, and $\boldsymbol{\Lambda}$ is diagonal.

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Note:

$$\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1} \;\Leftrightarrow\; \mathbf{A}\mathbf{X} = \mathbf{X}\mathbf{\Lambda}$$

thus, the jth column of $\mathbf{X}$ is an eigenvector of $\mathbf{A}$ and the jth entry of $\mathbf{\Lambda}$ is the corresponding eigenvalue.

Geometric Multiplicity

Given an eigenvalue $\lambda \in \mathbb{C}$.

- We denote E_λ the corresponding eigenspace
- E_λ is an $\mathbf{A}$-invariant subspace, i.e.,

$$\mathbf{A}E_\lambda \subseteq E_\lambda$$

- $\dim(E_\lambda)$ is the geometric multiplicity of λ
- Note:

$$\dim(E_\lambda) = \dim(\ker(\mathbf{A} - \lambda\mathbf{I}))$$

Characteristic Polynomial

Given $\mathbf{A} \in \mathbb{C}^{m \times m}$.

- We call

$$p_{\mathbf{A}}(z) = \det(z\mathbf{I} - \mathbf{A})$$

  the characteristic polynomial
- $\lambda \in \mathbb{C}$ is an eigenvalue of $\mathbf{A}$ iff $p_{\mathbf{A}}(\lambda) = 0$

$$\Rightarrow \text{Even if } \mathbf{A} \in \mathbb{R}^{m \times m}, \lambda \text{ may be complex!}$$

Algebraic multiplicity

- Fundamental theorem of algebra:

$$p_{\mathbf{A}}(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$$

  for some $\lambda_j \in \mathbb{C}$.
- The algebraic multiplicity of λ is its multiplicity as a root of $p_{\mathbf{A}}$
- If $\mathbf{A} \in \mathbb{C}^{m \times m}$, then $\mathbf{A}$ has m eigenvalues, counted with algebraic multiplicity.

$$\Rightarrow \text{Every matrix has at least one eigenvalue}$$

Similarity Transformations

Let $\mathbf{X} \in \mathbb{C}^{m \times m}$ be non-singular

- The map $\mathbf{A} \mapsto \mathbf{X}\mathbf{A}\mathbf{X}^{-1}$ is called a similarity transformation of $\mathbf{A}$
- Two matrices $\mathbf{A},\ \mathbf{B} \in \mathbb{C}^{m \times m}$ iff $\exists \mathbf{X}$ non-singular s.t.

$$\mathbf{B} = \mathbf{X}\mathbf{A}\mathbf{X}^{-1}$$

- Let $\mathbf{X}$ be non-singular, then $\mathbf{A}$ and $\mathbf{X}\mathbf{A}\mathbf{X}^{-1}$ have the same
  - i) characteristic polynomial
  - ii) eigenvalues
  - iii) algebraic and geometric multiplicities

Theorem:

The algebraic multiplicity of an eigenvalue λ is at least as great as its geometric multiplicity.

Diagonalizability

- $\mathbf{A}$ is called non-defective if the algebraic multiplicity equals the geometric multiplicity for all eigenvalues
- $\mathbf{A} \in \mathbb{C}^{m \times m}$ is non-defective iff it has an eigenvalue decomposition.

Computing Eigenvalues

Eigenvalues correspond to roots of a polynomial

There exists no closed form for roots of polynomials of degree ≥ 5

How do we compute the eigenvalues?

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We compute eigenvalue revealing decompositions:

- eigenvalue decomposition
- unitary eigenvalue decomposition
- Schur decomposition (Schur factorization)

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General procedure:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \xrightarrow{1.} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{2.} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \\ 0 & 0 & 0 & 0 & \times \end{bmatrix}$$

Phase 1: Direct computation

Phase 2: Iterative computation

Hessenberg Form

Q: Why compute the Hessenberg form?
$\Rightarrow$ "Can't we just use Householder like for linear systems?"

Hessenberg Form

Q: Why compute the Hessenberg form?
No, we cannot:

$$\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \xrightarrow{Q_1^* \cdot} \begin{bmatrix} \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix} \xrightarrow{\cdot Q_1} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}$$

Hessenberg Form

Q: Why compute the Hessenberg form?

(Upper) Hessenberg form is close to diagonal $\Rightarrow$ improves the scaling!

Hessenberg Form

Q: How do we compute the Hessenberg form?

Hessenberg Form

Q: How do we compute the Hessenberg form?

Householder!

$$\underset{\mathbf{A}}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_1^* \cdot} \underset{Q_1^* \mathbf{A}}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{\cdot Q_1} \underset{Q_1^* \mathbf{A} Q_1}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix}}$$

Hessenberg Form

Q: How do we compute the Hessenberg form?

$$\underset{Q_1^*\mathbf{A}Q_1}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{Q_2^*\cdot} \underset{Q_2^*Q_1^*\mathbf{A}Q_1}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix}} \xrightarrow{\cdot Q_2} \underset{Q_2^*Q_1^*\mathbf{A}Q_1Q_2}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix}}$$

Hessenberg Form

Q: How do we compute the Hessenberg form?

$$\underbrace{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix}}_{Q_2^*Q_1^*\mathbf{A}Q_1Q_2} \xrightarrow{Q_3^*\cdot} \underbrace{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}}_{Q_3^*Q_2^*Q_1^*\mathbf{A}Q_1Q_2} \xrightarrow{\cdot Q_2} \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

$$\underbrace{Q_3^*Q_2^*Q_1^*}_{=Q^*}\mathbf{A}\underbrace{Q_1Q_2Q_3}_{=Q}=H$$

Numerical methods

Power Iteration:

$\mathbf{v}^{(0)}$ some vector with $\|\mathbf{v}^{(0)}\| = 1$

for k = 1,2,...

$\mathbf{w} = \mathbf{A}\mathbf{v}^{(k-1)}$

$\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|$

$\lambda^{(k)} = \left(\mathbf{v}^{(k)}\right)^{\top}\mathbf{A}\mathbf{v}^{(k)}$

Numerical methods

Inverse Iteration:

$\mathbf{v}^{(0)}$ some vector with $\|\mathbf{v}^{(0)}\| = 1$

for k = 1,2,...

Solve $(\mathbf{A} - \mu\mathbf{I})\mathbf{w} = \mathbf{v}^{(k-1)}$

$\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|$

$\lambda^{(k)} = \left(\mathbf{v}^{(k)}\right)^\top \mathbf{A}\mathbf{v}^{(k)}$

Numerical methods

Rayleigh Quotient Iteration:

$\mathbf{v}^{(0)}$ some vector with $\|\mathbf{v}^{(0)}\| = 1$

$\lambda^{(0)} = \left(\mathbf{v}^{(0)}\right)^\top \mathbf{A}\mathbf{v}^{(0)}$

for k = 1,2,...

Solve $(\mathbf{A} - \lambda^{(k-1)}\mathbf{I})\mathbf{w} = \mathbf{v}^{(k-1)}$

$\mathbf{v}^{(k)} = \mathbf{w}/\|\mathbf{w}\|$

$\lambda^{(k)} = \left(\mathbf{v}^{(k)}\right)^\top \mathbf{A}\mathbf{v}^{(k)}$

Numerical methods

"Pure" QR algorithm (without shift):

$$\mathbf{A}^{(0)} = \mathbf{A}$$

for k = 1,2,...

$$\mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)}$$

$$\mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)}$$

Numerical methods

"Practical" QR algorithm (with shift):

$\left(\mathbf{Q}^{(0)}\right)^\top \mathbf{A}^{(0)} \mathbf{Q}^{(0)} = \mathbf{A}$

for k = 1,2,...

Pick $\mu^{(k)}$ e.g. $\mu^{(k)} = \mathbf{A}^{(k-1)}_{m,m}$

$\mathbf{Q}^{(k)} \mathbf{R}^{(k)} = \mathbf{A}^{(k-1)} - \mu^{(k)} \mathbf{I}$

$\mathbf{A}^{(k)} = \mathbf{R}^{(k)} \mathbf{Q}^{(k)} + \mu^{(k)} \mathbf{I}$

If any off-diag. element $\mathbf{A}_{j,j+1}$ us sufficiently small:

Set $\mathbf{A}_{j,j+1} = \mathbf{A}_{j+1,j} = 0$

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = \mathbf{A}^{(k)}$$

and apply the QR decomposition to $\mathbf{A}_1$ and $\mathbf{A}_2$.

Numerical methods

Jacobi

Bisection

$\vdots$