Sketching Eigenvalue Problems
Lecture 21

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Eigenvalue problem with sketching

- Let $\mathbf{A} \in \mathbb{R}^{m \times m}$ be symmetric
- Goal: Estimating (parts of) the spectrum via sketching
- Rayleigh-Ritz: We have seen that if $\mathbf{A} \in \mathbb{H}_n$ then

$$\lambda_{\min} = \min_{\|x\|=1} x^* \mathbf{A} x$$

- Common situation: m is way to large!
  but we have an idea "where" to look for the eigenvalues

$$\mathbf{B} \in \mathbb{R}^{m \times d}$$

  is an approximate space over which we seek to find eigenpairs

Rayleigh-Ritz (Galerkin projection)

- Given $\mathbf{B} = [\mathbf{b}_1|...|\mathbf{b}_d] \in \mathbb{R}^{m \times d}$
- Seeking eigenpairs in $\mathrm{Span}(\mathbf{b}_1, ..., \mathbf{b}_d)$, i.e.,

$$\mathbf{ABy} = \lambda \mathbf{By}$$

  with $\mathbf{ABy} - \lambda \mathbf{By} = \mathbf{r} \perp \mathrm{Span}(\mathbf{b}_1, ..., \mathbf{b}_d)$
- Find $\mathbf{y} \in \mathbb{R}^d$ and $\lambda \in \mathbb{C}$ s.t.

$$\mathbf{B}^\top(\mathbf{ABy} - \lambda \mathbf{By}) = \mathbf{0}$$

  NOTE: Here $\{\mathbf{b}_1, ..., \mathbf{b}_d\}$ are assumed to be orthonormal
- The above can be generalized to the eigenvalue problem

$$\mathbf{M}_* \mathbf{y} = \mathbf{B}^\dagger \mathbf{ABy} = \lambda \mathbf{y}$$

  where $\mathbf{y} \neq 0$ and $\lambda \in \mathbb{C}$

Rayleigh-Ritz

- Let $(\mathbf{y}_*, \lambda_*)$ be an eigenpair of $\mathbf{M}_*$. Then

$$\|\mathbf{ABy}_* - \lambda\mathbf{By}_*\|_2 = \|(\mathbf{AB} - \mathbf{BM}_*)\mathbf{y}_*\|_2$$

- This yields[1]

$$\min_{\mathbf{M}\in\mathbb{C}^{d\times d}} \|\mathbf{AB} - \mathbf{BM}\|_F$$

[1] Thm. 11.4.2 in Parlett *The Symmetric Eigenvalue Problem* (1998)

Rayleigh-Ritz – algorithmically

- We need to construct $\mathbf{B}$
- Krylov subspace approach:
  $\mathbf{r} \leftarrow$ Initial residual
  $\mathbf{b}_1 = \mathbf{r}/\|\mathbf{r}\|$
  For $p = 1$ to $d - 1$:
  $\quad \mathbf{w} = (\mathbf{I} - \mathbf{B}_{p-1}\mathbf{B}^*_{p-1})\mathbf{b}_{p-1}$
  $\quad \mathbf{b}_p = \mathbf{w}/\|\mathbf{w}\|$
  $\quad \mathbf{B}_p = [\mathbf{B}_{p-1}|\mathbf{b}_p]$
- Then apply QR algorithm to compute the spectrum:
  $\mathbf{A}^{(0)} = \mathbf{B}^\top \mathbf{A}\mathbf{B}$
  for $k = 1, 2, ...$
  $\quad \mathbf{Q}^{(k)}\mathbf{R}^{(k)} = \mathbf{A}^{(k-1)}$
  $\quad \mathbf{A}^{(k)} = \mathbf{R}^{(k)}\mathbf{Q}^{(k)}$

Let's put this to the test ...

Computational setup:

- Consider the 2D Laplacian approximated using finite differences
- Recall 1D-Laplacian

$$\mathbf{L}_1 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

- Then in 2D

$$\mathbf{L}_1 = \mathbf{L}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{L}_1$$

Sketching Rayleigh-Ritz

- We want to sketch

$$\min_{\mathbf{M}\in\mathbb{C}^{d\times d}} \|\mathbf{AB}-\mathbf{BM}\|_F$$

- Consider the sketch $\mathbf{S}\in\mathbb{C}^{s\times m}$ then

$$\min_{\mathbf{M}\in\mathbb{C}^{d\times d}} \|\mathbf{S}(\mathbf{AB}-\mathbf{BM})\|_F \tag{1}$$

- Sketching Rayleigh-Ritz (sRR) then finds $\hat{\mathbf{M}}_*$ minimizing (1), i.e.,

$$\mathbf{M}_* = (\mathbf{SB})^\dagger \mathbf{SAB}$$

- $s = 4d$ results in distortion $\varepsilon = 1/\sqrt{2}$ for the range of $[\mathbf{AB}, \mathbf{B}]$.

Let's try to make this better I

- Apply different sketches:
  i) SRTT:
  $\mathbf{G} \in \mathbb{R}^{n \times m}$
  $\boldsymbol{\sigma} \in \mathbb{R}^n$ i.i.d. Rademacher
  $\mathbf{i} \in \mathbb{R}^d$ random selection index
  $\mathbf{F} = \boldsymbol{\sigma} .* \mathbf{G}$
  $\mathbf{F} = \mathrm{dct}(\mathbf{F})$ discrete cosine transform (column wise applied)
  $\mathbf{F} = \mathbf{F}[\mathbf{i}, :]/\sqrt{n/d}$
  ii) SSE:
  $\zeta = 8$
  $\mathbf{G} \in \mathbb{R}^{n \times m}$
  For $i = 1$ to m
  $\quad \mathbf{i} \in \mathbb{R}^{\zeta}$ random selection index
  $\quad \boldsymbol{\sigma} \in \mathbb{R}^d$ i.i.d. Rademacher
  $\quad \mathbf{S}[\mathbf{i}, i] = \boldsymbol{\sigma}/\sqrt{\zeta}$

Let's try to make this better II

- Remove the Pseudo potential:
  Note that $[\mathbf{U}, \mathbf{T}] = \mathrm{qr}(\mathbf{SB})$ yields

$$\mathbf{M}_* = \mathbf{T}^{-1}\mathbf{U}^*\mathbf{SAB}$$

- How to invert T?
  $\rightarrow$ Back substitution
- $\mathbf{U} \in \mathbb{R}^{m \times n}$ with $\mathbf{U}_{ij} = 0$ for $i > j$ and $m \geq n$
- Solving $\mathbf{Ux} = \mathbf{b}$ from "bottom up yields"

$$\begin{aligned}
\mathbf{x}_n &= \mathbf{b}_n / \mathbf{U}_{n,n} \\
\mathbf{x}_{n-1} &= (\mathbf{b}_{n-1} - \mathbf{U}_{n-1,n} * \mathbf{x}_n) / \mathbf{U}_{n-1,n-1} \\
&\vdots \\
\mathbf{x}_i &= (\mathbf{b}_i - \mathbf{U}_{i,i+1:n} * \mathbf{x}_{i+1:n}) / \mathbf{U}_{i,i}
\end{aligned}$$