MATLAB Review
– Numerical Integration –
Lecture 4

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Middle Riemann Sums

Let $f : [a, b] \to \mathbb{R}$ be a function defines on a closed interval $[a, b] \subset \mathbb{R}$ and let $\{x_0, ..., x_n\}$ be a partition of $[a, b]$, i.e.,

$$a = x_0 < x_1 < ... < x_n = b.$$

Then

$$R(f, n) = \sum_{i=1}^{n} f(\tilde{x}_i)\Delta x_i$$

where $\Delta x_i = x_i - x_{i-1}$ and $\tilde{x}_i = (x_i + x_{i+1})/2$.

Middle Riemann sum error

- Let $f : [a, b] \to \mathbb{R}$ be a twice continuous differentiable function and

$$M = \sup_{x \in [a,b]} |f''(x)|$$

Then

$$|R_{\text{mid}}(f, n) - F| \leq \frac{M(b-a)^3}{24n^2} \sim \mathcal{O}\left(\frac{1}{n^2}\right)$$

Simpson's rule

We are interested in computing

$$F = \int_a^b f(x)\ dx$$

Simpson's rule:
Let $f : [a, b] \to \mathbb{R}$ be a function defines on a closed interval $[a, b] \subset \mathbb{R}$ and let $\{x_0, ..., x_n\}$ be a partition of $[a, b]$ with n even, i.e.,

$$a = x_0 < x_1 < ... < x_n = b.$$

Then

$$S(f, n) = \frac{\Delta x}{3} \left(f(x_0) + 4 \sum_{i=0}^{n/2-1} f(x_{2i+1}) + 2 \sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n) \right)$$

Simpson's rule error

- Let $f : [a, b] \to \mathbb{R}$ be a four-times continuously differentiable function and

$$M = \sup_{x \in [a,b]} |f''(x)|$$

Then

$$|S(f, n) - F| \leq \frac{M(b-a)^5}{180n^4} \sim \mathcal{O}\left(\frac{1}{n^4}\right)$$

Monte Carlo Estimator

We approximate F by averaging samples of the function f at uniform random points in $[a, b]$.

Formally: Given N uniform random variables $X_i \sim \mathcal{U}(a, b)$, its PDF is

$$\rho(x) = \frac{1}{b-a}\mathbb{1}_{[a,b]}(x)$$

and define the Monte Carlo estimator as

$$\left\langle F^N \right\rangle = (b-a)\frac{1}{N}\sum_{i=1}^{N} f(X_i)$$