Homework assignment

Due Date: 03/17 by 11:59 pm

Fill in your solutions in the Pluto notebook provided below. Once completed, execute the notebook and export it as a PDF. Upload the PDF to Gradescope for grading, and assign the pages to the respective exercises. Please ensure that all of your solutions, including the code you wrote, are visible and legible in the exported PDF before submitting it to Gradescope. Keep in mind that adjustments to assignments after the submission deadline will not be accommodated.

Conceptual Problems

Exercise 1:

Prove that

$$[a_p^\dagger, a_q]_+ = \delta_{p,q}.$$

Solution:

Your solution goes here ...

Exercise 2:

Compute

$$\langle -|a_j a_i a_n a_p^\dagger a_q^\dagger a_r a_s a_m^\dagger a_l^\dagger a_k^\dagger|-\rangle$$

using Wick's theorem for the absolute vacuum state.

Solution:

Your solution goes here ...

Exercise 3:

Consider the Hamiltoian

$$H = U\left(n_{1,\uparrow}n_{1,\downarrow} + n_{2,\uparrow}n_{2,\downarrow}\right) + t\left(a_{1,\uparrow}^\dagger a_{2,\uparrow} + a_{2,\uparrow}^\dagger a_{1,\uparrow} + a_{1,\downarrow}^\dagger a_{2,\downarrow} + a_{2,\downarrow}^\dagger a_{1,\downarrow}\right)$$

where

$$n_{i,\sigma} = a_{i,\sigma}^\dagger a_{i,\sigma}$$

Analytically determine all solutions to the Schrödinger equation fulfilling

$$(n_{1,\uparrow} + n_{1,\downarrow} + n_{2,\uparrow} + n_{2,\downarrow})|\Psi\rangle = 2|\Psi\rangle$$

Solution:

Your solution goes here ...

Programming Problems

Exercise 4:

Write two functions that generate $a_{p,s}$ and $a_{p,s}^\dagger$ given $K \in \mathbb{N}$, $p \in [\![K]\!]$ and $s \in \{\frac{1}{2}, -\frac{1}{2}\}$.

```
# Your code goes here
```

Verify your code by checking the CAR for the generated matrices $a_{p,s}$ and $a_{p,s}^\dagger$.

```
# Your code goes here
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Exercise 5:

Write a function that generates the Hamiltonian

$$H = U \sum_{i=1}^{K} n_{i,\uparrow} n_{i,\downarrow} + t \sum_{\sigma \in \{\uparrow,\downarrow\}} \sum_{\substack{i,j=1 \\ \langle i,j \rangle = 1}}^{K} (a_{i,\sigma}^\dagger a_{j,\sigma} + c.\,c.)$$

provided the input U, t, and n.

```
# Your code goes here
```

Verify your code by checking your analytic solution to Exercise 3.

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# Your code goes here
```

Exercise 6 (Metal to Mott Insulator at Half-Filling):

At half-filling (one electron per site), the system undergoes a phase transition as U increases:

For $U = 0$ (the non-Interacting Limit): The system is a Luttinger liquid (gapless metallic state) with a well-defined Fermi surface.

For $U > 0$ but small: The system remains metallic, but correlation effects modify the low-energy behavior.

For $U \gg t$ (Strong Coupling Limit): The system enters a Mott insulating phase where charge transport is suppressed due to the energy cost of double occupancy. A charge gap opens, preventing conductivity.

Show this phase transition numerically.

```
# Your code goes here
```