3.) LEGENDRE'S EQ'N:

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$$

$$\alpha = \text{CONST}$$

HERE $p(x) = -\frac{2x}{1-x^2}$, $q(x) = \frac{\alpha(\alpha+1)}{1-x^2}$

$x \neq \pm 1$ ARE ORDINARY PTS.

TYPICALLY INTERESTED IN INTERVAL $-1 \leq x \leq 1$

$\Rightarrow$ SERIES AROUND $x = 0$:

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\Rightarrow (1-x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty} a_n n x^{n-1} +$$

$$+ \alpha(\alpha+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} a_n n(n-1)x^n -$$

CAN CHANGE TO $n=0$

$$- \sum_{n=1}^{\infty} 2a_n n x^n + \sum_{n=0}^{\infty} \alpha(\alpha+1) a_n x^n = 0$$

CAN CHANGE TO $n=0$

$$\Rightarrow \sum_{n=0}^{\infty} \left[a_{n+2}(n+2)(n+1) - \left(n(n+1) - \alpha(\alpha+1)\right) a_n \right] x^n =$$

$$= 0$$

$$\Rightarrow \quad a_{n+2} = \frac{n(n+1) - \alpha(\alpha+1)}{(n+2)(n+1)} a_n$$

FOR $\alpha \neq 0, 1, 2, 3$ THIS GIVES 2 SOL'NS

$$a_0 \left(1 + b_1 x^2 + b_2 x^4 + \dots\right)$$

$$a_1 \left(x + \nu_1 x^3 + \nu_2 x^5 + \dots\right)$$

CONV'G ON $-1 \leq x \leq 1$.

THIS FOLLOWS FROM THEORY, BUT ALSO FROM RATIO TEST:

E.G. WHEN IS

$$\lim_{n \to \infty} \left| \frac{b_{n+1} x^{2(n+1)}}{b_n x^{2n}} \right| < 1$$

i.e., WHEN IS

$$x^2 \lim_{n \to \infty} \left| \frac{b_{n+1}}{b_n} \right| < 1 \quad ?$$

B/c $\lim_{n\to\infty} \left| \frac{2n(2n+1)-\alpha(\alpha+1)}{(2n+2)(2n+1)} \right| = 1$

$\Rightarrow x^2 < 1 \Rightarrow |x| < 1.$

FOR $\alpha = 0, 1, 2, 3, \ldots$ ONE SERIES TERMINATES

$\Rightarrow$ LEGENDRE POLYNOMIAL

2. Legendre polynomials are defined by the formulas

$$P_0(x) = 1, \qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n(x^2-1)^n}{dx^n}, \qquad n = 1, 2, \ldots$$

(i) For $m < n$, by repeatedly integrating by parts show that

$$\int_{-1}^{1} P_n(x)\, x^m\, dx = 0,$$

and conclude that

$$\int_{-1}^{1} P_n(x) P_m(x)\, dx = 0 \quad \text{if} \quad n \neq m.$$

(ii) Again by repeatedly integrating by parts, show that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(1-x)^n(1+x)^n\, dx,$$

and by repeatedly integrating by parts yet again show that

$$\int_{-1}^{1}(1-x)^n(1+x)^n\, dx = \frac{2^{2n+1}(n!)^2}{(2n)!(2n+1)}.$$

Conclude that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

(iii) Evaluate the n-th derivative of $(1-x)^n(1+x)^n$ using the product rule and set $x = 1$ to conclude that $P_n(1) = 1$.

(iv) Show that

$$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} Q_n(x) u^n, \tag{1}$$

for some n-th order polynomials $Q_n(x)$, by expanding the left-hand side in the binomial series in powers of $(u^2 - 2xu)$, using the binomial formula to expand these powers, and rearranging the terms. (Of course, this is only valid for $|u| < 1$ and $|x| \leq 1$.)

2:) $P_0(x) = 1$, $P_n(x) = \frac{1}{2^n n!} \frac{d^n (x^2-1)^n}{dx^n}$, $n = 1, 2, \dots$

(i) $m < n$:

$$2^n n! \int_{-1}^{1} P_n(x)\, x^m \, dx = \underbrace{\frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\, x^m \Big|_{-1}^{1}}_{0} - m \int_{-1}^{1} \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}}\, x^{m-1}\, dx =$$

$$= \underbrace{-m \frac{d^{n-2}(x^2-1)^n}{dx^{n-2}}\, x^{m-1} \Big|_{-1}^{1}}_{0} + m(m-1) \int_{-1}^{1} \frac{d^{n-2}(x^2-1)^n}{dx^{n-2}}\, x^{m-2}\, dx$$

$$= (-1)^m m! \int_{-1}^{1} \frac{d^{n-m}(x^2-1)}{dx^{n-m}}\, dx =$$

$$= (-1)^m m! \frac{d^{n-m-1}(x^2-1)}{dx^{n-m-1}} \Big|_{-1}^{1} = 0$$

THE INTEGRATED PARTS VANISH BECAUSE

$$\frac{d^k (x^2-1)}{dx^k} = \sum_{j=0}^{k} \binom{k}{j} \frac{d^j (x-1)^n}{dx^j} \frac{d^{k-j}(x+1)^n}{dx^{k-j}} =$$

$$= \sum_{j=0}^{k} \binom{k}{j} \frac{n!}{j!} (x-1)^{n-j} \frac{n!}{(k-j)!} (x+1)^{n-k+j}$$

AND BOTH $n-j > 0$, $n-k+j > 0$ IF $k < n$.

$\Rightarrow \int_{-1}^{1} P_n(x) P_m(x)\, dx = 0$ IF $m < n \Rightarrow$ IF $m \neq n$.

(ii) $$\int_{-1}^{1} P_n^2(x)\,dx = \frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\left[\frac{d^n(x^2-1)^n}{dx^n}\right]^2 dx =$$

$$= \frac{(-1)^k}{2^{2n}(n!)^2}\int_{-1}^{1}\frac{d^{n-k}(x^2-1)^n}{dx^{n-1}}\;\frac{d^{n+k}(x^2-1)^n}{dx^{n+k}}\,dx =$$

$$= \frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\;\frac{d^{2n}x^{2n}}{dx^{2n}}\,dx =$$

$$= \frac{(2n)!}{2^{2n}\,n!}\int_{-1}^{1}(1-x^2)^n dx = \frac{(2n!)}{2^{2n}\,n!}\int_{-1}^{1}(1-x)^n(1+x)^n\,dx$$

$$\int_{-1}^{1}(1-x)^n(1+x)^n\,dx = \cancelto{0}{\frac{(1+x)^{n+1}}{n+1}(1-x)^n\Big|_{-1}^{1}} +$$

$$+\frac{n}{n+1}\int_{-1}^{1}(1+x)^{n+1}(1-x)^{n-1}\,dx =$$

$$= \frac{n(n-1)}{(n+1)(n+2)}\int_{-1}^{1}(1+x)^{n+2}(1-x)^{n-2}\,dx = \cdots$$

$$= \frac{(n!)^2}{(2n)!}\int_{-1}^{1}(1+x)^{2n}\,dx = \frac{(n!)^2}{(2n)!}\,\frac{1}{2n+1}\,(1+x)^{2n+1}\Big|_{-1}^{1} =$$

$$= \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

$$\Rightarrow \int_{-1}^{1} P_n^2(x)\,dx = \frac{(2n)!}{2^{2n}(n!)^2}\,\frac{2^{2n+1}(n!)^2}{(2n+1)!} = \frac{2}{2n+1}$$

(iii) $$\frac{d^n\,(1-x)^n(1+x)^n}{dx^n} =$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{n!}{j!} (1-x)^{n-j} \frac{n!}{(n-j)!} (x+1)^{n-n+j}$$

$$= \sum_{j=0}^{n} (-1)^j\, n! \binom{n}{j}^2 (1-x)^{n-j} (x+1)^j =$$

$$= n!\,(-1)^n \binom{n}{0}^2 (x+1)^n + (1-x)\left[\,\cdots\,\right]$$

AT $x=1$

$$\left.\frac{d^n\,(1-x)^n(1+x)^n}{dx^n}\right|_{x=1} = 2^n\, n!\, (-1)^n$$

$$\Rightarrow P_n(1) = \frac{1}{2^n n!}(-1)^n\left[2^n n!\,(-1)^n\right] = 1$$

(iv) $$\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} (u^2 - 2xu)^k =$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} u^k (u-2x)^k =$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} u^k \sum_{j=0}^{k} \binom{k}{j} u^j (-1)^{k-j} 2^{k-j} x^{k-j} =$$

$$= \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} 2^{k-j} x^{k-j} u^{k+j}$$

NOW LET $n = k+j$ $\quad m = k-j$

$\Rightarrow k = \frac{1}{2}(n+m) \quad j = \frac{1}{2}(n-m)$

NOTE: m RANGES FROM n TO 0 OR 1 IN STEPS OF 2

$$\Rightarrow \frac{1}{\sqrt{1-2xu+u^2}} =$$

$$= \sum_{n=0}^{\infty} u^n \sum_{m=n,n-2,\dots\geq 0} \binom{-\frac{1}{2}}{\frac{1}{2}(n+m)} \binom{\frac{1}{2}(n+m)}{\frac{1}{2}(n-m)} (-1)^{\frac{1}{2}(n-m)} 2^{\frac{1}{2}(n-m)} x^{\frac{1}{2}(n-m)} =$$

$$= \sum_{n=0}^{\infty} u^n Q_n(x)$$