Class Presentation report: Spring 2025
The Clebsch-Gordan Coefficients
and their connection to spin states

Chanaka D. Mapa Mudiyanselage*, Fabian M. Faulstich

February 25, 2025

Presentation date: 02/25/2025

1 Clebsch-Gordan Coefficients

Clebsch-Gordan Coefficients are named of German mathematicians Alfred Clebsch and Paul Gordan

1.1 Definition

Clebsch-Gordan Coefficients are expansion coefficients of **total angular momentum eigenstates** in an uncoupled tensor product basis.

1.2 Total angular momentum

Let us denote total anugular momentum by J and then;

$$J = L + S \tag{1}$$

, where

- L - orbital angular momentum
- S - spin angular momentum

These coefficients arise when two angular (orbital or spin) are coupled.

For example:

Let's take (j_1, m_1) and (j_2, m_2) with $j_1 = j_2 = \frac{1}{2}$ (spin-half states),
where j_i's are angular momenta and m_i's are magnetic quantum number

Since, they are independent, we connect them with a tensor product,
so that there are four possible configurations.

Let's illustrate the definition;

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \begin{cases} |\frac{1}{2}\frac{1}{2}\rangle \otimes |\frac{1}{2}\frac{1}{2}\rangle & = |\uparrow\uparrow\rangle \\ |\frac{1}{2}\frac{1}{2}\rangle \otimes |\frac{1}{2} - \frac{1}{2}\rangle & = |\uparrow\downarrow\rangle \\ |\frac{1}{2} - \frac{1}{2}\rangle \otimes |\frac{1}{2}\frac{1}{2}\rangle & = |\downarrow\uparrow\rangle \\ |\frac{1}{2} - \frac{1}{2}\rangle \otimes |\frac{1}{2} - \frac{1}{2}\rangle & = |\downarrow\downarrow\rangle \end{cases} \tag{2}$$

These are in the uncouple tensor product basis which defines in the definition.
For the case of spin-half we usually take m_1 and m_2, $\uparrow$ for $\frac{1}{2}$ (spin-up) and $\downarrow$ for $-\frac{1}{2}$ (spin-down)

By rules of the Quantum mechanics, J can have values;

$$|j_1 - j_2| \leq J \leq |j_1 + j_2| \tag{3}$$

So, here J can be either 0 or 1

Then we have;

$$|JM>= \begin{cases} |00\rangle \\ |11\rangle \\ |10\rangle \\ |1-1\rangle \end{cases} \tag{4}$$

These are in the coupled basis.

If we ask what it the connection between these bases, the answer is Clebsch-Gordan coefficients.

Since both states are are normalized there might be a unitary transformation that connects them. For this particular case it can be represented as a 4×4 matrix where the entries are Clebsch-Gordan coefficients.Common Physics Notation: In most physics applications, the Clebsch-

Gordan coefficients are defined using a real and orthogonal transformation, making them real for standard angular momentum coupling.

Let's look at the mathematical version of the definition;

$$|JM>= \sum_{m_1,m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2|JM\rangle \tag{5}$$

We start with the state JM and insert a complete set of states in the product basis this Bra-Ket part (exterior product of the states) here are the clebsch-Gordon coefficients which now act as expansion coefficients of the product basis.

$\sum_{m_1,m_2} |j_1 m_1 j_2 m_2\rangle\langle j_1 m_1 j_2 m_2|$ is the product basis and $\langle j_1 m_1 j_2 m_2|JM\rangle$ are the Clebsch-Gordan coefficients.

For example:
If we start with the coupled state $|10\rangle$ we can expand this as

$$|10\rangle = c_1|\uparrow\uparrow\rangle + c_2|\uparrow\downarrow\rangle + c_3|\downarrow\uparrow\rangle + c_4|\downarrow\downarrow\rangle \tag{6}$$

The numerical values for c_1, c_2, c_3, c_4 are given buy the Clebsch-Gordan coefficients.

2 How do we work with Clebsch-Gordan coefficients?

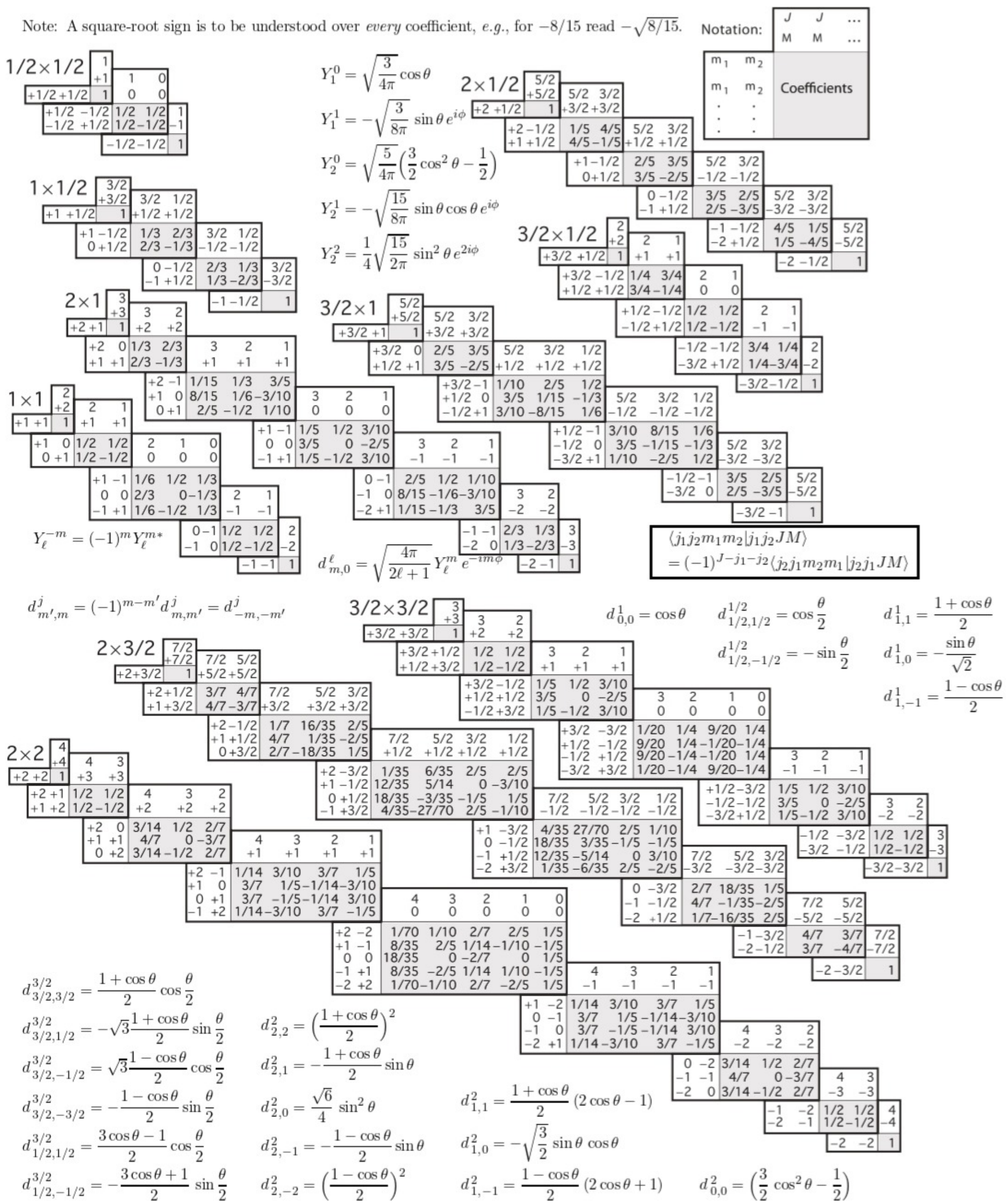
First, we don't have to derrive them every time we use them. Instead, when working on a problem we should always refer to some table of Clebsch-Gordan coefficients.

Note: We will discuss the derivation later in this report.

A great resource for Clebsch-Gordan coefficients is the;
Annual review by the particle data group

If we look at the notation table of the upper right corner, we see that uncoupled product states are written of the left of the table, coupled states are written on the top of the table and also that we always have to include a square root since this was omitted to improve readability.

Figure 1: Clebsch-Gordan Coefficients, Spherical Harmonics, and d Functions



2.1 Procedure to Find Clebsch-Gordan Coefficients

Four Steps to Find Clebsch-Gordan Coefficients

To determine a Clebsch-Gordan coefficient, follow these four steps:

1. **Identify** j_1 **and** j_2**:** Determine the two angular momenta being coupled. This helps in selecting the appropriate table.
2. **Identify** m_1 **and** m_2**:** Find the corresponding magnetic quantum numbers in the uncoupled basis. This helps in choosing the correct row in the table.
3. **Identify the total angular momentum** J **and** M**:** Locate the coupled basis state $|J, M\rangle$, which helps in selecting the correct column.
4. **Extract the coefficient:** Read the corresponding Clebsch-Gordan coefficient from the table. If a square root is involved, include it in the result. If there is a negative sign, place it outside the square root.

Examples

1) Suppose, we want to determine the Clebsch-Gordan coefficient of:

$$\langle \ \frac{3}{2} \quad \frac{1}{2} \quad 1 \quad 1 \ \mid \ \frac{5}{2} \quad \frac{3}{2} \ \rangle$$

Following the steps:

1. Use the $\big((j_1 =)\frac{3}{2}, (j_2 =)1\big)$ table
2. Locate the row corresponding to $\big((m_1 =)\frac{1}{2}, (m_2 =)1\big)$
3. Locate the column for the coupled state $\big((J =)\frac{5}{2}, (M =)\frac{3}{2}\big)$
4. Extract the coefficient: which is $\frac{3}{5}$

The result is $C = \sqrt{\frac{3}{5}}$

2) Suppose, we want to determine the Clebsch-Gordan coefficient of:

$$\langle \ 1 \quad -1 \quad 1 \quad 1 \ \mid \ 1 \quad 0 \ \rangle$$

Following the steps:

1. Use the $((j_1 =)1, (j_2 =)1)$ table
2. Locate the row corresponding to $((m_1 =) - 1, (m_2 =)1)$
3. Locate the column for the coupled state $((J =)1, (M =)0)$
4. Extract the coefficient: which is $-\frac{1}{2}$

The result is $C = -\sqrt{\frac{1}{2}}$

3) Suppose, we want to determine the Clebsch-Gordan coefficient of:

$$\langle \ 2 \quad -1 \quad 2 \quad 1 \ \mid \ 4 \quad 2 \ \rangle$$

$$\langle \ \frac{1}{2} \quad -\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \ \mid \ 0 \quad 0 \ \rangle$$

Following the steps:

1. Use the $((j_1 =)2, (j_2 =)2)$ table

2. Locate the row corresponding to $((m_1 =) - 1, (m_2 =)1)$
3. Locate the column for the coupled state $((J =)4, (M =)2)$ (Here, since we don't have such a column, look to the left and note that empty spaces are filled with zeros.)
4. Extract the coefficient: which is 0

The result is $C = 0$

Note:
Since Clebsch-Gordan tables form an orthogonal matrix, they allow for transformations in both directions:

- We can express a coupled state $|J, M\rangle$ in terms of product states $|j_1, m_1\rangle|j_2, m_2\rangle$.
- Similarly, we can express a product state in terms of coupled states.

This property ensures that Clebsch-Gordan coefficients serve as transformation coefficients between two equivalent bases in quantum mechanics.

3 Deriving Clebsch-Gordan Coefficients

Let's calculate Clebsch-Gordan Coefficients explicilty for the simple case of two spin-$\frac{1}{2}$ particle.

Note:
The derivation is recommended for understanding not for the real calculations, for the real calculations we should always refer a table or software.

3.1 Derivation

We consider two angular momenta j_1 and j_2, where in our case $j_1 = j_2 = \frac{1}{2}$. The magnetic quantum numbers m_1 and m_2 associated with these spins can take values:

$$m_1, m_2 \in \left\{-\frac{1}{2}, \frac{1}{2}\right\} \tag{7}$$

From the rules of angular momentum addition, the total angular momentum J satisfies:

$$|j_1 - j_2| \leq J \leq |j_1 + j_2| \tag{8}$$

Substituting $j_1 = j_2 = \frac{1}{2}$:

$$|\frac{1}{2} - \frac{1}{2}| \leq J \leq \frac{1}{2} + \frac{1}{2} \tag{9}$$

which simplifies to:

$$0 \leq J \leq 1 \tag{10}$$

Thus, J can take values 0 or 1.

The possible values of the total magnetic quantum number M are given by:

$$M = -J, -J + 1, \ldots, J \tag{11}$$

For $J = 0$:

$$M = 0 \tag{12}$$

For $J = 1$:

$$M \in \{-1, 0, 1\} \tag{13}$$

We also have the relation:

$$M = m_1 + m_2 \tag{14}$$

Since the total angular momentum J can take values 0 or 1, we have the following four states: $|00\rangle, |1-1\rangle, |10\rangle, |11\rangle$

The first number represents J, and the second one represents $M :\rightarrow |JM\rangle$

These states belong to the so-called *coupled basis*, since j_1 and j_2 couple to form J.

Compare this to the four states before coupling, where the first number stands for m_1 and the second one for $m_2 :\rightarrow |m_1 m_2\rangle \equiv |j_1 m_1 j_2 m_2\rangle \equiv |j_1 m_1 > \otimes |j_2 m_2\rangle$

Product basis: $\|m_1 m_2\rangle$	Coupled basis: $\|JM\rangle$	Coupled basis as a linear combination of the product basis
$\|+\frac{1}{2}+\frac{1}{2}\rangle$	$\|00\rangle$	$\alpha_1\|+\frac{1}{2}+\frac{1}{2}\rangle+\alpha_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\alpha_3\|-\frac{1}{2}+\frac{1}{2}\rangle+\alpha_4\|-\frac{1}{2}-\frac{1}{2}\rangle$
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	$\|1-1\rangle$	$\beta_1\|+\frac{1}{2}+\frac{1}{2}\rangle+\beta_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\beta_3\|-\frac{1}{2}+\frac{1}{2}\rangle+\beta_4\|-\frac{1}{2}-\frac{1}{2}\rangle$
$\|-\frac{1}{2}+\frac{1}{2}\rangle$	$\|10\rangle$	$\gamma_1\|+\frac{1}{2}+\frac{1}{2}\rangle+\gamma_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\gamma_3\|-\frac{1}{2}+\frac{1}{2}\rangle+\gamma_4\|-\frac{1}{2}-\frac{1}{2}\rangle$
$\|-\frac{1}{2}-\frac{1}{2}\rangle$	$\|11\rangle$	$\delta_1\|+\frac{1}{2}+\frac{1}{2}\rangle+\delta_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\delta_3\|-\frac{1}{2}+\frac{1}{2}\rangle+\delta_4\|-\frac{1}{2}-\frac{1}{2}\rangle$

Table 1: Relation between coupled and uncoupled bases

Note that these 16 coefficients are precisely the Clebsch-Gordan Coefficients

Before diving into the details, we can take steps to reduce the number of unknown coefficients. Since $m_1 + m_2$ must equal M, we immediately find links that:

$$|+\frac{1}{2}+\frac{1}{2}\rangle \leftrightarrow |11\rangle \tag{15}$$

$$|-\frac{1}{2}-\frac{1}{2}\rangle \leftrightarrow |1-1\rangle. \tag{16}$$

These correspondences uniquely determine two coefficients, reducing the number of unknowns by ten. (Terms corresponding to: $\alpha_1, \alpha_4, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_4, \delta_2, \delta_3, \delta_4$ will vanish)

Coupled basis: $\|JM\rangle$	Coupled basis as a linear combination of the product basis
$\|00\rangle$	$\alpha_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\alpha_3\|-\frac{1}{2}+\frac{1}{2}\rangle$
$\|1-1\rangle$	$\beta_4\|-\frac{1}{2}-\frac{1}{2}\rangle$
$\|10\rangle$	$\gamma_2\|+\frac{1}{2}-\frac{1}{2}\rangle+\gamma_3\|-\frac{1}{2}+\frac{1}{2}\rangle$
$\|11\rangle$	$\delta_1\|+\frac{1}{2}+\frac{1}{2}\rangle$

Table 2: Relation between coupled and uncoupled bases

However, for the remaining states, a unique one-to-one correspondence is not immediately apparent, so we assume a linear combination of both states:

$$|00\rangle = \alpha_2|+\frac{1}{2}-\frac{1}{2}\rangle + \alpha_3|-\frac{1}{2}+\frac{1}{2}\rangle \tag{17}$$

$$|10\rangle = \gamma_2|+\frac{1}{2}-\frac{1}{2}\rangle + \gamma_3|-\frac{1}{2}+\frac{1}{2}\rangle \tag{18}$$

Thus, only six coefficients remain to be determined.

These coefficients are not all independent.
Since we require the states to be normalized, we obtain the constraint:

$$\langle 00|00\rangle = 1 \quad \Rightarrow \quad |\alpha_2|^2 + |\alpha_3|^2 = 1 \tag{19}$$
$$\langle 1{-}1|1{-}1\rangle = 1 \quad \Rightarrow \quad |\beta_4|^2 = 1 \tag{20}$$
$$\langle 10|10\rangle = 1 \quad \Rightarrow \quad |\gamma_2|^2 + |\gamma_3|^2 = 1 \tag{21}$$
$$\langle 11|11\rangle = 1 \quad \Rightarrow \quad |\delta_1|^2 = 1 \tag{22}$$

Note:
Absolute square is must as we use complex numbers.

Additionally, since $|10\rangle$ and $|00\rangle$ must be orthogonal, we obtain another equation:

$$\langle 10|00\rangle = 0 \quad \Rightarrow \quad \gamma_2^*\alpha_2 + \gamma_3^*\alpha_3 = 0 \tag{23}$$

In order to determine these factors, apply the operator $\hat{J}^2$ on the states $|10\rangle$ and $|00\rangle$.

We can do this in two ways:
First, in the coupled basis, where we know that $J(J+1)$ is the correct eigenvalue.

$$\hat{J}^2|JM\rangle = J(J+1)|JM\rangle \quad \Rightarrow \quad \hat{J}^2|10\rangle = 2|10\rangle, \hat{J}^2|00\rangle = 0$$

And then in the product basis, by writing $\hat{J}$ as a vector sum of $\hat{j}_1$ and $\hat{j}_2$:

$$\hat{J} = \hat{j}_1 \otimes \mathbb{I}_2 + \hat{j}_2 \otimes \mathbb{I}_2 \tag{24}$$

By squaring this equation, we can express $\hat{J}^2$ only in terms of quantities that refer to the product basis:

$$\hat{J}^2 = \hat{j}_1^2 \otimes \mathbb{I}_2 + 2\hat{j}_1\hat{j}_2 + \mathbb{I}_2 \otimes \hat{j}_2^2 \tag{25}$$

Using ladder operators;

$$2\hat{j}_1\hat{j}_2 = j_{1+}j_{2-} + j_{1-}j_{2+}2j_{1z}j_{2z} \tag{26}$$

where the indexed number always refers to which spin we want to operate on. For example: j_{1-} is the lowering operator for the first number in a state.

Now we can have;

$$\begin{aligned}
\hat{J}^2|10\rangle &= \left[\hat{j}_1^2 + \hat{j}_2^2 + j_{1+}j_{2-} + j_{1-}j_{2+} + 2j_{1z}j_{2z}\right]\left[\gamma_2\left|+\frac{1}{2}-\frac{1}{2}\right\rangle + \gamma_3\left|-\frac{1}{2}+\frac{1}{2}\right\rangle\right] \\
&= \frac{3}{4}\gamma_2\left|+\frac{1}{2}-\frac{1}{2}\right\rangle + \frac{3}{4}\gamma_2\left|+\frac{1}{2}-\frac{1}{2}\right\rangle + \gamma_2\left|-\frac{1}{2}+\frac{1}{2}\right\rangle - \frac{1}{2}\gamma_2\left|+\frac{1}{2}-\frac{1}{2}\right\rangle \\
&+ \frac{3}{4}\gamma_3\left|-\frac{1}{2}+\frac{1}{2}\right\rangle + \frac{3}{4}\gamma_3\left|-\frac{1}{2}+\frac{1}{2}\right\rangle + \gamma_3\left|+\frac{1}{2}-\frac{1}{2}\right\rangle - \frac{1}{2}\gamma_3\left|-\frac{1}{2}+\frac{1}{2}\right\rangle \\
&= (\gamma_2+\gamma_3)\,| + \frac{1}{2} - \frac{1}{2}\rangle + (\gamma_2+\gamma_3)\,| - \frac{1}{2} + \frac{1}{2}\rangle \\
&= 2\gamma_2| + \frac{1}{2} - \frac{1}{2}\rangle + 2\gamma_3| - \frac{1}{2} + \frac{1}{2}\rangle
\end{aligned} \tag{27}$$

The last step, since we know this result must match the coupled basis.
Thus, $\gamma_2 = \gamma_3$;

$$\gamma_2 = \gamma_3 = \frac{1}{\sqrt{2}}. \tag{28}$$

For the state $|00\rangle$, a similar calculation yields:

$$\alpha_2 + \alpha_3 = 0 \tag{29}$$

Thus, we conclude:

$$\alpha_2 = -\alpha_3, \quad |\alpha_2| = |\alpha_3| = \frac{1}{\sqrt{2}}. \tag{30}$$

3.2 Determining Phases

The phase of these coefficients is determined by the Condon-Shortley phase convention, which states:
1). Ladder operators act such that their phase is positive and real ($\exp\{\iota\psi\} = +1$).

$$J_{\pm}|JM\rangle = \exp\{\iota\psi\}\sqrt{(J \mp M)(J \pm M + 1)}|JM \pm 1\rangle \tag{31}$$

2). The phase of the following Clebsch-Gordan coefficient are chosen to be positive and real.

$$\langle j_1 j_1 j_2 (J - j_1)|JJ\rangle \in \mathbb{R}, > 0 \tag{32}$$

From these conditions, we can determine all six phases assumed earlier.

For example, if both spins couple to total spin zero (i.e $J = 0$); the above mentioned Clebsch-Gordan coefficient corresponds to α_2:

$$\alpha_2(\in \mathbb{R}, > 0) = \langle\frac{1}{2} + \frac{1}{2}\frac{1}{2} - \frac{1}{2}|00\rangle\frac{1}{\sqrt{2}}, \quad \alpha_3 = -\frac{1}{\sqrt{2}} \tag{33}$$

Which can be uniquely determine as $\alpha_2 = \frac{1}{\sqrt{2}}$ which intern $\alpha_3 = -\frac{1}{\sqrt{2}}$
The final result can be obtained as;

$$\alpha_2 = -\alpha_3 = -\frac{1}{\sqrt{2}}, \beta_4 = 1, \gamma_2 = -\gamma_3 = \frac{1}{\sqrt{2}}, \delta_1 = 1$$

Thus, we fully determine the Clebsch-Gordan coefficients for two spin-$\frac{1}{2}$ particles.
These values are typically summarized in a table:

	$\|00\rangle$	$\|1-1\rangle$	$\|10\rangle$	$\|11\rangle$
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	α_1	β_1	γ_1	δ_1
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	α_2	β_2	γ_2	δ_2
$\|-\frac{1}{2}+\frac{1}{2}\rangle$	α_3	β_3	γ_3	δ_3
$\|-\frac{1}{2}-\frac{1}{2}\rangle$	α_4	β_4	γ_4	δ_4

Table 3: Table format for the coefficients

	$\|11\rangle$	$\|10\rangle$	$\|00\rangle$	$\|1-1\rangle$
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	1	0	0	0
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\|-\frac{1}{2}+\frac{1}{2}\rangle$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$\|-\frac{1}{2}-\frac{1}{2}\rangle$	0	0	0	1

Table 4: Updated Table with Specific Values

And most of the time, elements where the Clebsch-Gordan coefficient is zero are simply left out of the table, such that it takes on a strange, non-square shape.

	$\|11\rangle$	$\|10\rangle$	$\|00\rangle$	$\|1-1\rangle$
$\|+\frac{1}{2}-\frac{1}{2}\rangle$	1			
$\|+\frac{1}{2}-\frac{1}{2}\rangle$		$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	
$\|-\frac{1}{2}+\frac{1}{2}\rangle$		$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	
$\|-\frac{1}{2}-\frac{1}{2}\rangle$				1

Table 5: Usual table format