Distribution Theory: A Deep Dive into Generalized Functions

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Introduction

- We all at some point study function,limit, and derivative.
- In Differential Equations Solutions with sharp turns are interesting and Important.

  Therefore, we need something new, as Many real-world phenomena such as signal processing involves singularities:
- proposed new object called Distribution or Generalised functions

Brief Historical Example

- in 1927 Paul Dirac studied the differentiation of Heaviside Function, but there's a problem at zero.
  Definition:
  $$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

- **Paul Dirac (1930s):** Introduced the Dirac delta function in quantum mechanics.

graph of general function H'(x)=$\delta(x)$

- The classical derivative has a problem at $x = 0$

Dirac Delta (Impulse) Function

definition

$$\delta(x) = 0, \quad \text{for all } x \neq 0. \tag{1}$$

This means that the delta function is zero everywhere except at $x = 0$.

such that

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1. \tag{2}$$

Hence $\delta(x)$ is not a function in the classical sense instead, it is a distribution.

- we look at it action on test function.

Shifted Heaviside Function

Definition:

$$H(x-a) = \begin{cases} 0, & x < a \\ 1, & x \geq a \end{cases} \tag{3}$$

Test Functions in Distribution Theory

Definition:

$\phi : \mathbb{R} \to \mathbb{R}$

A **test function** is a smooth function with compact support.

$$C_c^\infty(\mathbb{R}). \tag{4}$$

Properties of Test Functions:

- differentiable arbitrarily often: $\phi(x) \in C^\infty(\mathbb{R})$.
- Compact support: There exists a bounded interval $[a, b]$ such that $\phi(x) = 0$ outside this region.
- $$\phi : \mathbb{R}^n \to \mathbb{R}$$
  :
- The space of test function
  $$\mathcal{D}(\mathbb{R}^n) = C_c^\infty(\mathbb{R}^n)$$

Action of Dirac delta on test functions

In the theory of distributions, a generalized function is considered not a function in itself but only through how it affects other functions when "integrated" against them.

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)dx = \phi(0) \tag{5}$$

Therefore dirac delta is a distribution that tell us what happens at a single point in the test function.

$$T : \phi \mapsto \mathbb{R}. \tag{6}$$

$$T(\phi) = \int_{\mathbb{R}} \delta(x)\phi(x)\, dx \in \mathbb{R}. \tag{7}$$

Shifted Dirac Delta Function

$$\int_{-\infty}^{\infty} \delta(x-a)\phi(x)dx = \phi(a) \tag{8}$$

Thank You!

Any Questions?

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