General Legendre Equation and Associated Legendre Polynomials

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Introduction to Legendre Functions

The Legendre differential equation is given by:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \quad \text{for} \quad -1 < x < 1$$

The solution can be expressed as a power series:

$$y(x) = \sum_{k=0}^{\infty} a_k x^k$$

with the recurrence relation:

$$a_{k+2} = \frac{k(k+1) - \lambda}{(k+1)(k+2)} a_k$$

Legendre Polynomials

When $\lambda = n(n+1)$, the series terminates, leading to Legendre polynomials $P_n(x)$:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[(x^2 - 1)^n\right]$$

The first few Legendre polynomials are:

n	$\lambda = n(n+1)$	$P_n(x)$
0	0	$P_0(x) = 1$
1	2	$P_1(x) = x$
2	6	$P_2(x) = \frac{1}{2}(3x^2 - 1)$
3	12	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$
4	20	$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

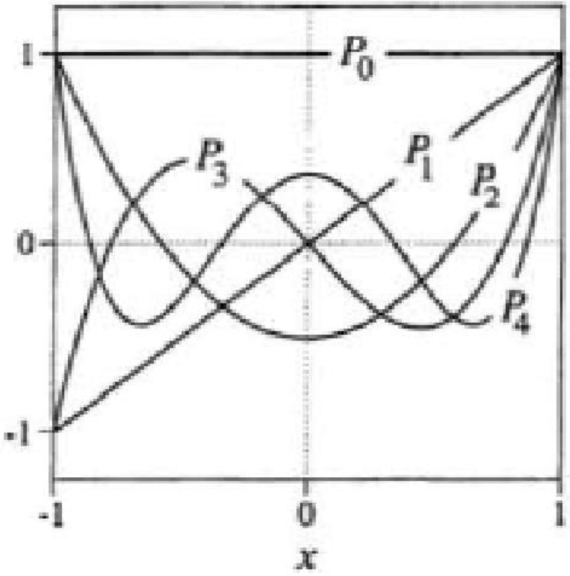

Figure: Plots of the Legendre polynomials P_1 to P_4

Orthogonality of Legendre Polynomials

Legendre polynomials are orthogonal over the interval $[-1, 1]$:

$$\int_{-1}^{1} P_j(x) P_k(x)\, dx = 0 \quad \text{for} \quad j \neq k$$

The norm of $P_n(x)$ is given by:

$$\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$$

Generating Function

The generating function for Legendre polynomials is:

$$(1 - 2xr + r^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) r^n \quad \text{for} \quad |x| \leq 1, |r| < 1$$

This function can be used to derive various properties of Legendre polynomials, such as recurrence relations:

$$nP_n(x) = (2n - 1)xP_{n-1}(x) - (n - 1)P_{n-2}(x)$$

Associated Legendre Polynomials

The associated Legendre equation is a generalization of the Legendre equation:

$$(1-x^2)y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0$$

The solutions are the associated Legendre polynomials $P_n^m(x)$:

$$P_n^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

These polynomials are used in solving problems with spherical symmetry, such as in quantum mechanics and electromagnetism.

Applications of Legendre Polynomials

- **Physics:** Used in solving Laplace's equation in spherical coordinates.
- **Quantum Mechanics:** Spherical harmonics, which involve associated Legendre polynomials, describe the angular part of the wave function.
- **Electromagnetism:** Multipole expansions of electric and magnetic fields.

Thank You!

Any Questions?