The Clebsch-Gordan Coefficients and their connection to spin states

DEPARTMENT OF MATHEMATICS| 02/25/25
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Outline

- Overview of Clebsch-Gordan Coefficients
- How do we work with Clebsch-Gordan Coefficients?
- Procedure to Find Clebsch-Gordan Coefficients
- Some Examples
- Derivation justification

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I. Overview

- **Definition:** Clebsch-Gordan Coefficients are the expansion coefficients of total angular momentum eigen-states in an uncoupled tensor product basis.
- **Let's talk about physics first:**

Couple two angular momenta:

Example:

$$|j_1 m_1\rangle \otimes |j_2 m_2\rangle = \begin{cases} |\frac{1}{2}\,\frac{1}{2}\rangle \otimes |\frac{1}{2}\,\frac{1}{2}\rangle & = |\uparrow\uparrow\rangle \\ |\frac{1}{2}\,\frac{1}{2}\rangle \otimes |\frac{1}{2} - \frac{1}{2}\rangle & = |\uparrow\downarrow\rangle \\ |\frac{1}{2} - \frac{1}{2}\rangle \otimes |\frac{1}{2}\,\frac{1}{2}\rangle & = |\downarrow\uparrow\rangle \\ |\frac{1}{2} - \frac{1}{2}\rangle \otimes |\frac{1}{2} - \frac{1}{2}\rangle & = |\downarrow\downarrow\rangle \end{cases}$$

$$|JM> = \begin{cases} |00\rangle \\ |11\rangle \\ |10\rangle \\ |1-1\rangle \end{cases}$$

I. Overview

- **Mathematical version of the definition:**

$$|JM >= \sum_{m_1,m_2} \overbrace{|j_1 m_1 j_2 m_2\rangle}\underbrace{\langle j_1 m_1 j_2 m_2|JM\rangle}$$

For example:

$$|10\rangle = c_1|\uparrow\uparrow\rangle + c_2|\uparrow\downarrow\rangle + c_3|\downarrow\uparrow\rangle + c_4|\downarrow\downarrow\rangle$$

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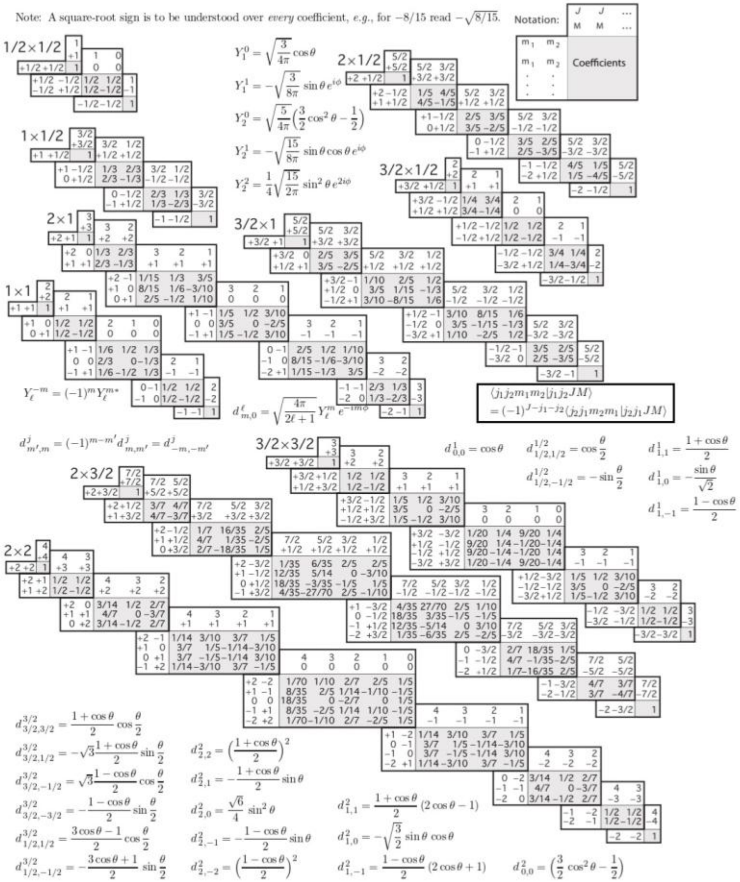
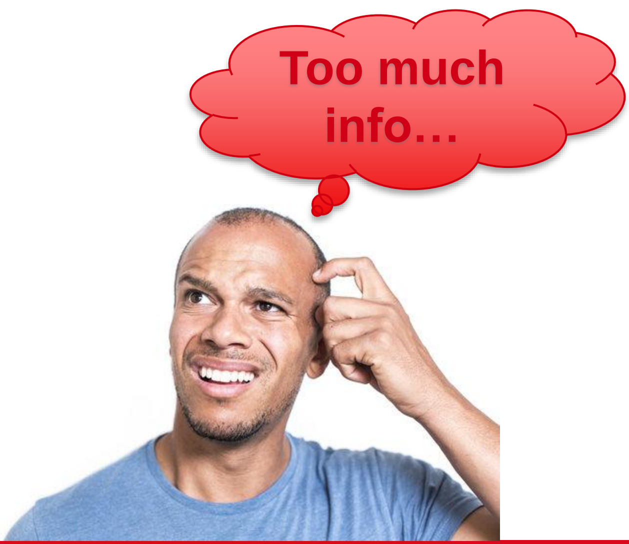
II. How do we work with them?

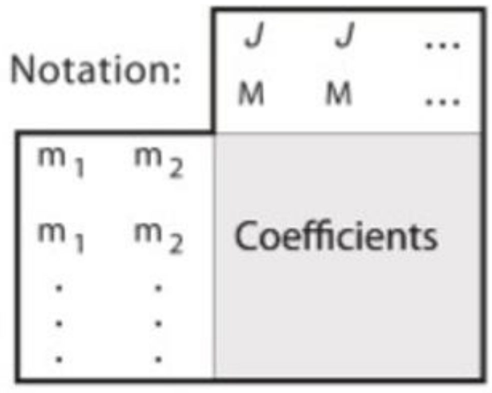
- **Don’t derive, refer to a table or software!!!**
- **A good resource: “Annual review by the particle data group”**





II. How do we work with them?

- **Don’t worry!!!... Let’s do this step by step…**



- **We always have to include a square root since this was omitted to improve readability**

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

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III. Procedure to Find C-G Coefficients

- **Four Steps…**

1. **Identify j1 and j2: Determine the …** ← **Table**

2. **Identify m1 and m2: Find the …** ← **Row**

   Column ↓

3. **Identify the total angular momentum J and M : Locate the …**

4. **Extract the coefficient: Read the …** ← **Value**

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IV. Examples – (i)

$$\left\langle \tfrac{3}{2}\ \tfrac{1}{2}\ \ 1\ \ 1 \ \middle|\ \tfrac{5}{2}\ \tfrac{3}{2} \right\rangle$$

Following the steps:

1.

2.

3.

4.

$3/2 \times 1$

	5/2
	+5/2
+3/2 +1	1

	5/2	3/2
	+3/2	+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

	5/2	3/2	1/2
	+1/2	+1/2	+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

	5/2	3/2	1/2
	−1/2	−1/2	−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

	5/2	3/2
	−3/2	−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

	5/2
	−5/2
−3/2 −1	1

IV. Examples – (ii)

$\langle\, 1 \quad -1 \quad 1 \quad 1 \mid 1 \quad 0 \,\rangle$

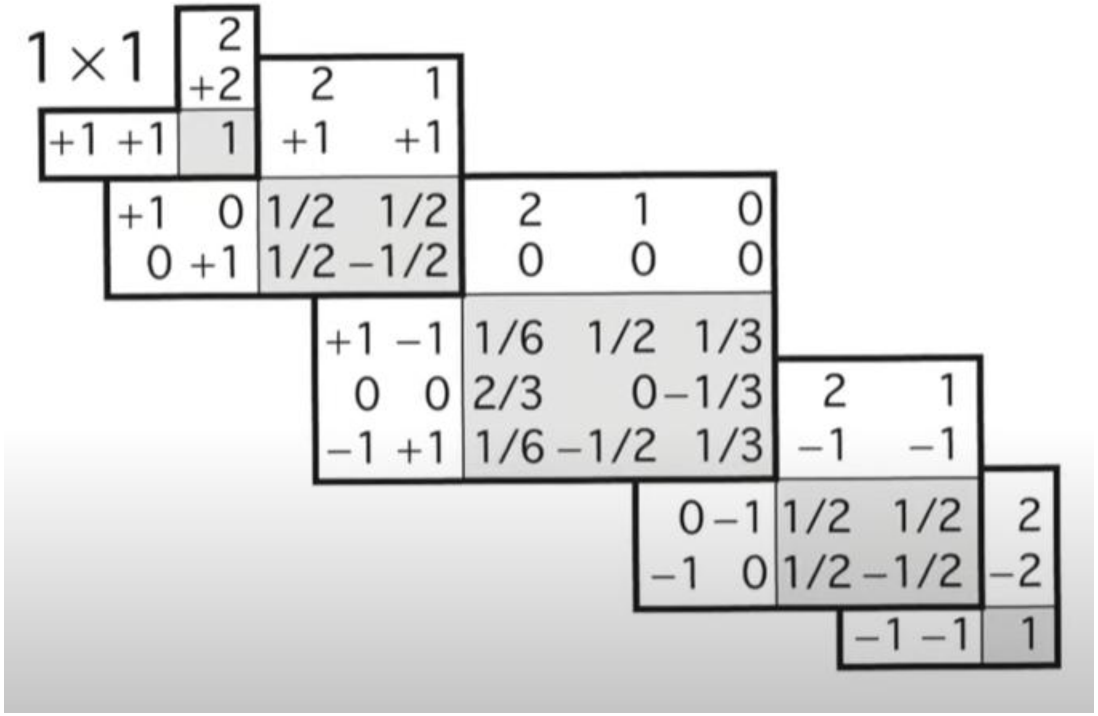
Following the steps:

1.

2.

3.

4.



IV. Examples – (iii)

$$\langle\, 2 \quad -1 \quad 2 \quad 1 \mid 4 \quad 2 \,\rangle$$

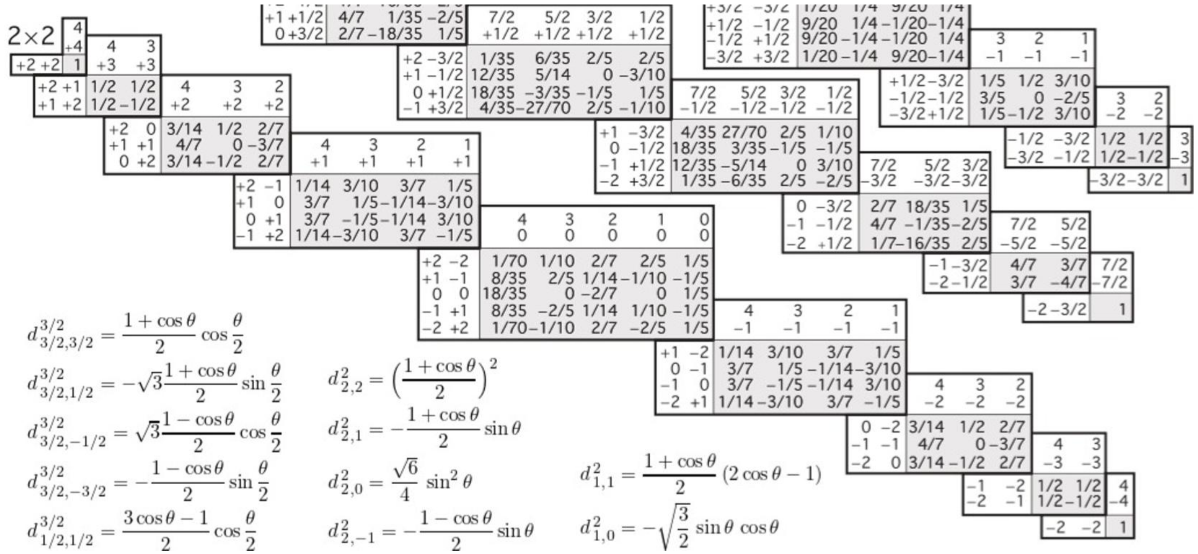
Following the steps:

1.

2.

3.

4.



$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

IV. Examples – (iv – spin one-half system)

$$\left\langle \tfrac{1}{2} \;\; -\tfrac{1}{2} \;\; \tfrac{1}{2} \;\; \tfrac{1}{2} \;\middle|\; 0 \;\; 0 \right\rangle$$

Following the steps:

1.

2.

3.

4.

1/2×1/2	1			
	+1	1	0	
+1/2 +1/2	1	0	0	
+1/2 −1/2		1/2	1/2	1
−1/2 +1/2		1/2	−1/2	−1
−1/2 −1/2				1

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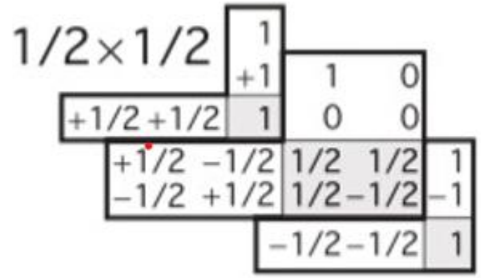
V. Derivation justification

Derived

	$\lvert 11\rangle$	$\lvert 10\rangle$	$\lvert 00\rangle$	$\lvert 1-1\rangle$
$\lvert +\frac{1}{2} - \frac{1}{2}\rangle$	1	0	0	0
$\lvert +\frac{1}{2} - \frac{1}{2}\rangle$	0	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0
$\lvert -\frac{1}{2} + \frac{1}{2}\rangle$	0	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	0
$\lvert -\frac{1}{2} - \frac{1}{2}\rangle$	0	0	0	1

Table 4: Updated Table with Specific Values

Annual review by the particle data group



Note: **Since Clebsch-Gordan tables form an orthogonal matrix, they allow for transformations in both directions**

ANY QUESTIONS!!!

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Thank you!