$$\frac{d}{dx_1}\left(2\alpha(r_1 - x_1)\exp(-\alpha(r-x)^2)\right)$$

$$\frac{d^2}{dx_1^2} f(x) = \left[\frac{d}{dx_1}\left(2\alpha(r_1 - x_1)\right)\right]\exp(-\alpha(r-x)^2)$$

$$+ 2\alpha(r_1 - x_1)\frac{d}{dx}\left(\exp(-\alpha(r-x)^2)\right)$$

$$= -2\alpha\exp(-\alpha(r-x)^2) + 4\alpha^2(r_1-x_1)^2\exp(-\alpha(r-x)^2)$$

$$= \exp(-\alpha(r-x)^2)\left[4\alpha^2(r_1-x_1)^2 - 2\alpha\right]$$

$\Rightarrow$ Hermite polynomial of $(4\alpha^2(r_1-x_1)^2 - 2\alpha$

since

$\Rightarrow$ $H_2(x) = 4x^2 - 2$

Comparing both we get

$$x = \alpha^{1/2}(r_1 - x_1)$$

$$\Rightarrow \alpha^{2/2} H_2(\alpha^{1/2}(r_1-x_1))\exp(-\alpha(r-x)^2)$$

If we generalize this for n

$$\frac{d^n}{dx_1^n} f(x) = \alpha^{n/2} H_n(\alpha^{1/2}(r_1-x_1))\exp(-\alpha(r-x)^2)$$

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$$\exp\left(-\frac{\alpha\beta|x-y|^2}{\gamma}\right)\sum D_{ir}^{m} E_{js}^{n} F_{kt}^{p}\, \Lambda_{mnp}^{r,z}(\gamma)$$

$$\exp\left(-\frac{\alpha\beta|x-y|^2}{\gamma}\right)\sum D_{ir}^{m} E_{js}^{n} F_{kt}^{p}\, \frac{\partial^m}{\partial z_1^m}\frac{\partial^n}{\partial z_2^n}\frac{\partial^p}{\partial z_3^p}\exp\left(-\gamma|r-z|^2\right)$$

$$\exp\left(\frac{-\alpha\beta|x-y|^2}{\gamma}\right)\sum D_{ir}^{m} E_{js}^{n} F_{kt}^{p}\, \frac{\partial^m}{\partial z_1^m}\frac{\partial^n}{\partial z_2^n}\left(\gamma^{p/2}H_p\left(\sqrt{\gamma}\,(r_3-z_3)\right)\right)\exp\left(-\gamma|r-z|^2\right)$$

$$\exp\left(\frac{-\alpha\beta|x-y|^2}{\gamma}\right)\sum D_{ir}^{m} E_{js}^{n} F_{kt}^{p}\, \frac{\partial^m}{\partial z_1^m}\left(\gamma^{n/2}H_n\left(\sqrt{\gamma}\,(r_2-z_2)\right)\right)\left(\gamma^{p/2}H_p\left(\sqrt{\gamma}\,(r_3-z_3)\right)\right)\exp(\quad)$$

$$\exp\left(\frac{-\alpha\beta|x-y|^2}{\alpha+\beta}\right)\sum D_{ir}^{m} E_{js}^{n} F_{kt}^{p}\left(\gamma^{m/2}H_m\left(\sqrt{\gamma}\,(r_1-z_1)\right)\right)\left(\gamma^{n/2}H_n\left(\sqrt{\gamma}\,(r_2-z_2)\right)\right)\left(\gamma^{p/2}H_p\left(\sqrt{\gamma}\,(r_3-z_3)\right)\right)$$
$$\exp(\quad)$$

$$\exp\left(\frac{-\alpha\beta|x-y|^2}{\gamma}\right)\sum_m D_{ir}^{m}\gamma^{m/2}H_m\left(\sqrt{\gamma}\,(r_1-z_1)\right)\sum_n E_{js}^{n}\gamma^{n/2}H_n\left(\sqrt{\gamma}\,(r_2-z_2)\right)$$
$$\sum_p F_{kt}^{p}\gamma^{p/2}H_p\left(\sqrt{\gamma}\,(r_3-z_3)\right)\quad \exp\left(-\gamma|r-z|^2\right)$$

$$\sum_m D_{ir}^{m}\gamma^{m/2}H_m\left(\sqrt{\gamma}\,(r_1-z_1)\right) = (r_1-x_1)^i(r_1-y_1)^r$$

$$\sum_n E_{js}^{n}\gamma^{n/2}H_n\left(\sqrt{\gamma}\,(r_2-z_2)\right) = (r_2-x_2)^j(r_2-y_2)^s$$

$$\sum_p F_{kt}^{p}\gamma^{p/2}H_p\left(\sqrt{\gamma}\,(r_3-z_3)\right) = (r_3-x_3)^k(r_3-y_3)^t$$

$$\exp\left(-\alpha(r-x)^2\right)\ \exp\left(-\beta(r-y)^2\right)$$

$$\exp\left(-\alpha(r-x)^2+(\beta(r-y)^2)\right)$$

$$\exp\left(-\alpha r^2-\alpha x^2+2\alpha r x-\beta r^2-\beta y^2+2\beta r y\right)$$

$$\exp\left(-r^2(\alpha+\beta)+2r(\alpha x+\beta y)-\alpha x^2-\beta y^2\right)$$

$$\exp\left[-(\alpha+\beta)\left(r^2-\frac{2r(\alpha x+\beta y)}{\alpha+\beta}+\frac{\alpha x^2+\beta y^2}{\alpha+\beta}\right)\right]$$

$$\exp\left[-(\alpha+\beta)\left(r^2-\frac{2r(\alpha x+\beta y)}{\alpha+\beta}+\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)^2-\left(\frac{\alpha x+\beta y}{\alpha+\beta}\right)^2+\frac{\alpha x^2+\beta y^2}{\alpha+\beta}\right)\right]$$

$$\exp\left[-(\alpha+\beta)\left(\left(r-\frac{\alpha x+\beta y}{\alpha+\beta}\right)^2-\frac{\alpha^2x^2+\beta^2y^2+2\alpha x\beta y}{(\alpha+\beta)^2}+\frac{\alpha x^2+\beta y^2}{\alpha+\beta}\right)\right]$$

$$\exp\left[-(\alpha+\beta)\left(r-\frac{\alpha x+\beta y}{\alpha+\beta}\right)^2+\frac{\cancel{\alpha^2x^2}+\cancel{\beta^2y^2}+2\alpha\beta xy-\cancel{\alpha^2x^2}-\alpha\beta y^2-\alpha\beta x^2-\cancel{\beta^2y^2}}{(\alpha+\beta)}\right]$$

$$\exp\left[-(\alpha+\beta)\left(r-\frac{\alpha x+\beta y}{\alpha+\beta}\right)^2+\frac{(-\alpha\beta)(-2xy+x^2+y^2)}{\alpha+\beta}\right]$$

$$\exp\left[-\mu(r-z)^2\right]\exp\left(-\frac{\alpha\beta|x-y|^2}{\mu}\right)\qquad \therefore\quad \mu=\alpha+\beta,\quad z=\frac{\alpha x+\beta y}{\alpha+\beta}$$

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$$\int dr\, \Lambda^{\gamma, z}_{mnp}(r) = \frac{\partial^m}{\partial z_1^m} \frac{\partial^n}{\partial z_2^n} \frac{\partial^p}{\partial z_3^p} \int dr \exp(-\gamma(r-z)^2)$$

$$= \delta_{m0}\,\delta_{n0}\,\delta_{p0} \left(\frac{\pi}{\gamma}\right)^{3/2}$$

~~Let~~

Consider

$$I = \int \exp(-\gamma(r'-z)^2)\,dr'$$

$$I^2 = \left(\int \exp(-\gamma(r'-z)^2)\,dr'\right)\left(\int \exp(-\gamma(s-z)^2)\,ds\right)$$

$$= \iint \exp(-\gamma(r'-z)^2 - \gamma(s-z)^2)\,dr'ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-\gamma((r'-z)^2 + (s-z)^2))\,dr'ds$$

Let $r'-z = r\cos\theta$ and $s-z = r\sin\theta$

$$\Rightarrow (r'-z)^2 + (s-z)^2 = r^2$$

$$= \int_0^{2\pi}\int_0^{\infty} \exp(-\gamma r^2)\, r\,dr\,d\theta$$

$$= \frac{-1}{2\gamma}\int_0^{2\pi} \Big[\exp(-\gamma r^2)\Big]_0^{\infty} d\theta$$

$$= \frac{-1}{2\gamma}\int_0^{2\pi} (-1)\,d\theta = \frac{1}{2\gamma}(2\pi) = \frac{\pi}{\gamma}$$

$$I^2 = \frac{\pi}{\gamma}$$

$$I = \sqrt{\frac{\pi}{\gamma}}$$

$$(r_1 - x_1)(r_1 - x_1)^{i-1}(r_1 - y_1)^{r}$$

$$= (r_1 - x_1) \sum_{m}^{i+r-1} D^{m}_{i-1,r} \kappa^{m/2} H_m\left(\sqrt{\kappa}(r_1 - z_1)\right)$$

$$= (r_1 - z_1) + (z_1 - x_1) \sum_{m}^{i+r-1} D^{m}_{i-1} \kappa^{m/2} H_m\left(\sqrt{\kappa}(r_1 - z_1)\right)$$

$$= \sum_{m}^{i+r-1} D^{m}_{i-1} \kappa^{m/2} \left((r_1 - z_1) + (z_1 - x_1)\right) H_m\left(\sqrt{\kappa}(r_1 - z_1)\right)$$

$$\because \kappa^{m/2}(r_1 - z_1) H_m\left(\sqrt{\kappa}(r_1 - z_1)\right) = \kappa^{\frac{(m-1)}{2}}\left(m H_{m-1} - \frac{1}{2} H_{m+1}\right)$$

$$\frac{d^3}{dx_1^3} f(x) = \alpha^{3/2} H_3(\alpha^{\frac{1}{2}}(r_1 - x_1)) \exp(-\alpha(r-x)^2)$$

$$\frac{d^2}{dx_1^2} f(x) = \alpha H_2(\alpha^{\frac{1}{2}}(r_1 - x_1)) \exp(-\alpha(r-x)^2)$$

$$\frac{d}{dx_1} f(x) = \alpha^{\frac{1}{2}} H_1(\alpha^{\frac{1}{2}}(r_1 - x_1) \exp(-\alpha(r-x)^2)$$

$H_1 = 2x$ comparing with $(2\alpha^{\frac{1}{2}}(r_1 - x_1))\alpha^{\frac{1}{2}}$

$$\Rightarrow \quad x = \alpha^{\frac{1}{2}}(r_1 - x_1)$$

$$\Rightarrow \quad H_1(\alpha^{\frac{1}{2}}(r_1 - x_1)) = 2\alpha^{\frac{1}{2}}(r_1 - x_1)$$

$$H_2(\alpha^{\frac{1}{2}}(r_1 - x_1)) = 4(\alpha^{\frac{1}{2}}(r_1 - x_1))^2 - 2$$

Since $H_3(x) = 8x^3 - 12x$

$$H_3(\alpha^{\frac{1}{2}}(r_1 - x_1)) = 8(\alpha^{\frac{1}{2}}(r_1 - x_1))^3 - 12(\alpha^{\frac{1}{2}}(r_1 - x_1))$$

$$= 8\alpha^{3/2}(r_1 - x_1)^3 - 12\alpha^{\frac{1}{2}}(r_1 - x_1)$$

Now $2H_1 - \frac{1}{2}H_3 = 4\alpha^{\frac{1}{2}}(r_1 - x_1) + 4\alpha^{\frac{3}{2}}(r_1 - x_1)^3 - 6\alpha^{\frac{1}{2}}(r_1 - x_1)$

$$\alpha^{\frac{1}{2}}(2H_1 - \tfrac{1}{2}H_3) = \alpha^{\frac{1}{2}}\left[\alpha^{\frac{1}{2}}(r_1 - x_1)\left[4 - 6 + 4\alpha(r_1 - x_1)^2\right]\right]$$

$$= \alpha(r_1 - x_1)\left[4(\alpha^{\frac{1}{2}}(r_1 - x_1))^2 - 2\right]$$

$$= \alpha(r_1 - x_1) H_2(\alpha^{\frac{1}{2}}(r_1 - x_1))$$

$$D^0_{01} = \cancel{\frac{1}{2\mu} D^{-1}_{00}}^{\to 0} + (Y_1 - X_1) D^0_{00} + \cancel{D^1_{00}}^{\to 0}$$

$$\boxed{D^0_{01} = (Y_1 - X_1)} \qquad \because D^0_{00} = 1$$

$$D^0_{11} = \cancel{\frac{1}{2\mu} D^{-1}_{0,1}}^{\to 0} + (Y_1 - X_1) D^0_{01} + D^1_{01}$$

$$= (Y_1 - X_1)^2 + \frac{1}{2\mu}$$

$$D^1_{01} = \frac{1}{2\mu} D^0_{00} + \cancel{(Y_1 - X_1) D^1_{0,0}}^{\to 0} + \cancel{D^2_{00}}^{\to 0}$$

$$D^1_{01} = \frac{1}{2\mu}$$

$$\nabla (fg) = f\nabla g + g\nabla f$$

$$\nabla^2 (fg) = \nabla f \nabla g + f\nabla^2 g + \nabla g \nabla f + g\nabla^2 f$$

$$= 2\nabla g \nabla f + f\nabla^2 g + g\nabla^2 f$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

$$H_2(x) = e^{x^2} \frac{d^2}{dx^2} e^{-x^2}$$

$$= e^{x^2} \frac{d}{dx} \left(e^{-x^2} (-2x) \right)$$

$$= e^{x^2} \left(-2\left(-2x\, e^{-x^2} (x) + e^{-x^2}\right)\right)$$

$$= e^{x^2} \left(-4x^2 e^{-x^2} \mp 2e^{-x^2} \right)$$

$$= +4x^2 - 2$$

$$\int f(x)\, f(y)\, f(z)\, dx\, dy\, dz = \int f(x)\, dx \int f(y)\, dy \int f(z)\, dz$$

$$\exp(-\gamma\, |\overbrace{r - z}^{r}|^2)$$

$$= \exp(-\gamma\, (x^2 + y^2 + z^2))$$

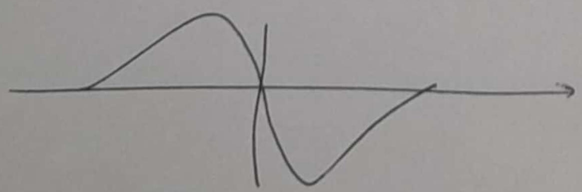
$$\cancel{\frac{\partial}{\partial z^?} \int \exp(-\gamma z^2)\, dz}$$

$$\frac{\partial}{\partial z^?} \int \exp(-\gamma (x - z)^2)\, dx$$

$$\int \partial_z \exp(-\gamma (x - z)^2)\, dx = \int -2\gamma (x - z) \exp(-\gamma (x - z)^2)\, dx$$

$$= -2\gamma \int \underset{\substack{\uparrow \\ \text{antisym}}}{u}\ \underbrace{\exp(-\gamma u^2)}_{\text{symmetric}}\, du$$

$\cancel{Z\,k}\,\gamma z^2$



$$f \qquad du$$

$$-2\gamma \int u \left[-2\gamma u \exp(-\gamma u^2)\right) du \qquad \overset{2g}{\int f G} = FG - \int F g$$

$$\downarrow$$

$$\underbrace{-2\gamma\, u \exp(-\gamma u^2)\Big|_{-\infty}^{\infty}}_{=0} + 2\gamma \underbrace{\int \exp(-\gamma u^2)\, du}_{=\sqrt{\frac{\pi}{\gamma}}}$$

$$u\int u\exp(\quad)\,du - \int\left(\frac{d}{du}(u)\int u\exp(\quad)\,du\right)du$$

$$-2\gamma\left[\int u\,\frac{d}{du}\exp(-\gamma u^2)\,du\right] = -2\gamma\left[\cancel{u\exp(-\gamma u^2)}\Big|_{-\infty}^{\infty}\right.$$

$$+\int \exp(-\gamma u^2)\,du$$

$$= +2\gamma\left(-\sqrt{\frac{\pi}{\gamma}}\right) = -2\sqrt{\gamma\pi}$$

Kinetic

Laplacian of cartesian Gaussian

$$\nabla^2 C_{ijk}^{\alpha x}(r) = \left(\nabla^2 P_{ijk}^{x}(r)\right)\exp(-\alpha(r-x)^2)$$
$$+ P_{ijk}^{x}(r)\,\nabla^2 \exp(-\alpha(r-x)^2) \quad \leftarrow (*)$$
$$+ 2\nabla P_{ijk}^{x}(r) \cdot \nabla \exp(-\alpha(r-x)^2)$$

$$\nabla^2 P_{ijk}^{x}(r) = \frac{\partial^2}{\partial r_1^2}(r_1-x_1)^i (r_2-x_2)^j (r_3-x_3)^k$$

$$= -i\left(-(i-1)\right)(r_1-x_1)^{i-2}(r_2-x_2)^j (r_3-x_3)^k$$

$$= i(i-1)(r_1-x_1)^{i-2}(r_2-x_2)^j (r_3-x_3)^k$$

$$\nabla^2 P_{ijk}^{x}(r) = i(i-1)\, P_{i-2\,jk}^{x}(r) \quad \leftarrow (1)$$

$$\frac{\partial^2}{\partial x_1^2}\exp(-\alpha(r-x)^2) = \frac{\partial}{\partial r_1}\left[\left(\exp(-\alpha(r-x)^2)(-2\alpha(r_1-x_1))\right)\right]$$

$$= \exp(-\alpha(r-x)^2)(-2\alpha) + (-2\alpha(r_1-x_1))^2 \exp(-\alpha(r-x)^2)$$

$$= -2\alpha \exp(-\alpha(r-x)^2) + 4\alpha^2 (r_1-x_1)^2 \exp(-\alpha(r-x)^2)$$

$\leftarrow (2)$

$$\frac{\partial}{\partial r_1} P^x_{ijk}(r) = \frac{\partial}{\partial r_1} (r_1 - x_1)^i (r_2 - x_2)^j (r_3 - x_3)^k$$

$$= i (r_1 - x_1)^{i-1} (r_2 - x_2)^j (r_3 - x_3)^k$$

$$\frac{\partial}{\partial r_1} \exp(-\alpha (r-x)^2) = -2\alpha (r_1 - x_1) \exp(-\alpha (r-x)^2)$$

$$2 \frac{\partial}{\partial r_1} P^x_{ijk}(r) \frac{\partial}{\partial r_1} \exp(-\alpha (r-x)^2) = 4i\alpha P^x_{i-1jk}(r) \exp(-\alpha (r-x)^2)(r_1 - x_1)$$

$$= 4i\alpha P^x_{ijk}(r) \exp(-\alpha (r-x)^2) \quad \leftarrow 3$$

using 1, 2 & 3 in A

we get

$$= i(i-1) C^{\alpha,x}_{i-2jk}(r) + 4\alpha^2 C^{\alpha,x}_{i+2jk}(r) - 2\alpha(2i+1) C^{\alpha,x}_{ijk}(r)$$

$$\Rightarrow \quad V(r) = \sigma^2 \int_0^{2\pi}\int_0^{\pi} \frac{R^2 \sin\theta}{\sqrt{r^2+R^2-2rR\cos\theta}}\, d\theta\, d\phi \qquad \vec{r}\,' = (r', \theta', \phi')$$

Let $u = r^2 + R^2 - 2rR\cos\theta$

$du = +2rR\sin\theta\, d\theta$

$\sin\theta\, d\theta = \frac{1}{2rR}\, du$

$$= \frac{\cancel{2}\pi\sigma^2 R^{\cancel{2}}}{\cancel{2}r\cancel{R}} \int_{(r-R)^2}^{(r+R)^2} \frac{1}{\sqrt{u}}\, du$$

$$= \frac{2\pi\sigma^2}{r}\left[u^{\frac{1}{2}}\right]_{(r-R)^2}^{(r+R)^2}$$

---

error function

$$\operatorname{erf} = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\, dt$$

① $\operatorname{erf}(-x) = -\operatorname{erf}(x)$

② $-1 \le \operatorname{erf}(x) \le 1$

continous and differentiable for all $\mathbb{R}$

$$\int_0^{\sqrt{\pi}\, r} e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{\pi}\, r)$$

Gaussian Integral

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$= 4\pi \int_0^r dR \exp(-\gamma R^2) \frac{\sigma R^3}{r} + 4\pi \int_r^\infty \exp(-\gamma R^2) \sigma R \, dR$$

$$\int_r^\infty \exp(-\gamma R^2) \sigma R \, dR$$

Let $u = \gamma R^2$

$\Rightarrow du = \gamma 2 R \, dR$

$\frac{1}{2\gamma} du = R \, dR$

$$\frac{d}{dR}\left(\exp(-\gamma R^2)\right)$$

$$-2\gamma R \exp(-\gamma R^2)$$

$$+\frac{\sigma}{2\gamma} \int_{+\gamma r^2}^\infty \exp^{(-u)} du$$

$$\frac{\sigma}{2\gamma} \left[\frac{\exp(-u)}{-1} \right]_{\gamma r^2}^\infty$$

$$\boxed{\cancel{\frac{\sigma}{2\gamma} \exp(-\gamma r^2)}}$$

$$\frac{\sigma}{r} \int_0^r \exp(-\gamma R^2) R^2 \, dR = \frac{\sigma}{r} \int_0^r \underbrace{R}_{u} \underbrace{\exp(-\gamma R^2) R}_{v} \, dR$$

$$= \frac{\sigma}{r} \left[R \int_0^r R \exp(-\gamma R^2) dR + \int_0^r \frac{\exp(-\gamma R^2)}{2\gamma} dR \right.$$

$$= \frac{\sigma}{r} \left[\cancel{r\left(\frac{\exp(-\gamma r^2)}{-2\gamma} \right)} + \int_0^r \frac{\exp(-\gamma R^2)}{2\gamma} dR \right]$$

$$x R^2 = t^2$$

$$t = \sqrt{x}\, R \qquad \Big| \quad 2xR\,dR = 2t\,dt$$

$$\frac{xR\,dR}{t} = dt$$

$$t \to 0 \qquad \frac{x\cancel{R}\,dR}{\sqrt{x}\,\cancel{R}} =$$

$$t \to \sqrt{x}\, r$$

error function

$$\frac{dR}{x} = dt$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

$$\int_0^{\sqrt{x}\, r} e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2} \operatorname{erf}(\sqrt{x}\, r)$$

$$\frac{\cancel{2}\pi}{x^{3/2} r} \cdot \frac{\sqrt{\pi}}{\cancel{2}} = \frac{(\pi)^{3/2}}{x^{3/2} r} \operatorname{erf}(\sqrt{x}\, r)$$

$$\frac{\partial^m}{\partial z_1^m}\frac{\partial^n}{\partial z_2^n}\frac{\partial^p}{\partial z_3^p}\int dr\, \Lambda_{000}^{\kappa,z}(r) = \frac{2\pi}{\kappa} F_0\left(\kappa\,|r-z|^2\right) \triangleq \frac{2\pi}{\kappa} R_{mnp0}$$

Calculating R_{mnp0} is efficient with the recursive relations

$$R_{00pj} = Z_3 R_{00p-1j+1} + (p-1) R_{00p-2j+1}$$

$$R_{0npj} = Z_2 R_{0n-1pj+1} + (n-1) R_{0n-2pj+1}$$

$$R_{mnpj} = Z_1 R_{m-1npj+1} + (m-1) R_{m-2npj+1}$$

$$R_{000j} = (-2\kappa)^j F_j\left(\kappa\,|r-z|^2\right)$$

where $F_j(T) = \Gamma(m)\left(\bar{\Gamma}(m,T)\right) / 2T^m$ with $m = j + \frac{1}{2}$

$$R_{mnpj} = 0 \quad \text{if} \quad m, n, p < 0$$

$$R_{0000} = F_0\left(\kappa\,(r-z)^2\right)$$

$$R_{0010} = Z_3 R_{0001} + \cancel{(0) R_{00-11}}$$

$$R_{0010} = Z_3 F_1\left(\kappa\,(r-z)^2\right)$$

$$R_{1110} = Z_1 R_{0111} + 0$$

$$R_{0110} = Z_2 R_{0011} + 0$$

$$R_{0011} = Z_3 R_{0002} = Z_3 (-2\kappa)^2 F_2\left(\kappa\,(r-z)^2\right)$$

$$R_{0110} = Z_2 Z_3 (-2\kappa)^2 F_2\left(\kappa\,(r-z)^2\right)$$

$\cancel{R_{1110}}$

$$R_{0111} = Z_2 R_{0012} + 0$$

$$R_{0012} =$$

$\cancel{R_{0012}}$

$\cancel{R_{0012} = Z_3 R_{0003}}$

$\cancel{R_{001j}} = Z_3 R_{000j}$

$\cancel{R_{000}}$

$$R_{001j} = Z_3 R_{000j+1} = Z_3 (-2\chi)^{j+1} F_{j+1}(\chi|r-z|^2)$$

$$R_{002j} = Z_3 R_{001j+1} + R_{000j+1}$$

$$\Rightarrow R_{00pj} = Z_3 R_{00p-1j+1} + (p-1) R_{00p-2j+1}$$

$$R_{0npj} = Z_2 R_{00pj+1} + (n-1) \cancel{R_{0n-2pj+1}}$$

$$R_{01pj} = Z_2 (Z_3 R_{00p-1j+2} + (p-1) R_{00p-2j+2})$$